MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
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Homework Sheet 7
Model Solutions

Compulsory questions

1 *Solve the following recurrence relations. i.e. find an explicit formula for a_n in terms of n, and prove that it works. [You do not need to prove that your formula works if the equation is a second order constant-coefficient homogeneous linear recurrence.]*

(a) $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = -1$, $a_1 = 2$.

We start by looking for solutions of the form $a_n = t^n$. This gives $t^n = 4t^{n-1} - 4t^{n-2}$, and so $t^2 = 4t - 4$, or $t^2 - 4t + 4 = 0$, i.e. $(t-2)^2 = 0$, so there is only one solution – $t = 2$. This means that $a_n = 2^n$ and $a_n = n2^n$ should both satisty $a_n = 4a_{n-1} - 4a_{n-2}$. We can cheack this: $2^n = 4\times 2^{n-1} - 4\times 2^{n-2}$ and $n\times 2^n = 4(n-1)\times 2^{n-1} - 4(n-2)\times 2^{n-2}$ both hold. Now we need to find A and B so that $a_n = A2^n + Bn2^n$ satisfies $a_0 = -1$, $a_1 = 2$. This gives the equations:

$$A + 0\times B = -1 \quad (1)$$
$$2A + 2B = 2 \quad (2)$$

These are easily solved to get $A = -1$, $B = 2$, so the solution is $a_n = n2^{n+1} - 2^n$.

(b) $a_n = 2a_{n-1} + 3(n-2)$, $a_0 = 1$. *[Hint: try subtracting a_n from 2^n]*

We start by looking at the first few terms:

n	a_n	$2^n - a_n$
0	1	0
1	-1	3
2	-2	6
3	-1	9
4	4	12
5	17	15
6	46	18

This suggests that $a_n = 2^n - 3n$ is the solution. We prove this by induction. When $n = 0$, we have already checked that this base case holds.

Now suppose that $a_n = 2^n - 3n$. We want to show that $a_{n+1} = 2^{n+1} - 3(n+1)$. By the recurrence relation, $a_{n+1} = 2a_n + 3(n-1)$. By our induction hypothesis, this is equal to $2(2^n - 3n) + 3(n-1) = 2^{n+1} - 6n + 3n - 3 = 2^{n+1} - 3(n+1)$ as required. Therefore, the formula works for all n by induction.

(c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 5$, $a_1 = 3$.

We start by looking for solutions of the form $a_n = t^n$. We get $t^n = t^{n-1} + 3t^{n-2}$, and thus $t^2 - t - 3 = 0$, so $t = \frac{1 \pm \sqrt{13}}{2}$ are the solutions. We now just need to find A and B so that $A\left(\frac{1+\sqrt{13}}{2}\right)^0 + B\left(\frac{1-\sqrt{13}}{2}\right)^0 = 5$ and $A\left(\frac{1+\sqrt{13}}{2}\right)^1 + B\left(\frac{1-\sqrt{13}}{2}\right)^1 = 3$. This gives

$$A + B = 5 \tag{3}$$

$$A\left(\frac{1+\sqrt{13}}{2}\right) + B\left(\frac{1-\sqrt{13}}{2}\right) = 3 \tag{4}$$

We subtract $\left(\frac{1-\sqrt{13}}{2}\right)$ times the first equation from the second to get $\sqrt{13}A = 3 - 5\left(\frac{1-\sqrt{13}}{2}\right) = \frac{1}{2} + \frac{5\sqrt{13}}{2}$, or $A = \frac{1}{2\sqrt{13}} + \frac{5}{2}$. From this we get $B = \frac{5}{2} - \frac{1}{2\sqrt{13}}$. Therefore, the general solution is

$$a_n = \frac{5}{2}\left(\left(\frac{1+\sqrt{13}}{2}\right)^n + \left(\frac{1-\sqrt{13}}{2}\right)^n\right) + \frac{1}{2\sqrt{13}}\left(\left(\frac{1+\sqrt{13}}{2}\right)^n - \left(\frac{1-\sqrt{13}}{2}\right)^n\right)$$

(d) $a_n = \frac{1}{1+a_{n-1}}$, $a_0 = 1$. *[You may use F_n to denote the nth Fibonacci number in your formula.]*

We look at the first few values:

n	a_n
0	$\frac{1}{1}$
1	$\frac{1}{2}$
2	$\frac{2}{3}$
3	$\frac{3}{5}$

This leads us to conjecture that $a_n = \frac{F_{n+1}}{F_{n+2}}$. We prove this by induction. We have already checked the base case. Now suppose that $a_n = \frac{F_{n+1}}{F_{n+2}}$. We want to show that $a_{n+1} = F_{n+2}F_{n+3}$. We have that $a_{n+1} = \frac{1}{1+a_n}$ By our induction hypothesis, $1 + a_n = 1 + \frac{F_{n+1}}{F_{n+2}} = \frac{F_{n+2}+F_{n+1}}{F_{n+2}} = \frac{F_{n+3}}{F_{n+2}}$ Therefore, $a_{n+1} = \frac{F_{n+2}}{F_{n+3}}$ as required, so by induction, the formula holds for all n.

2 *Let F be a function defined by $F(0) = 1$, and*

$$F(n) = \begin{cases} F\left(\frac{n}{2}\right) & \textit{if } n \textit{ is even.} \\ F(n-1) + 2 & \textit{if } n \textit{ is odd.} \end{cases}$$

Prove that $F(n)$ is odd for all natural numbers n.

We prove this by strong induction on n. If $n = 0$, then by definition $F(0)$ is odd. Now suppose that $F(k)$ is odd for all $k < n$. We want to show that $F(n)$ is odd. If n is even, and $n > 0$, then $n \geqslant 2$, so $\frac{n}{2} < n$, so $F\left(\frac{n}{2}\right)$ is odd by our induction hypothesis, and therefore, $F(n)$ is odd. On the other hand, if n is odd, then $n-1 < n$, so $F(n-1)$ is odd by our inductive hypothesis, so $F(n)$ is also odd, since an odd number plus 2 is also odd. Therefore, by strong induction, $F(n)$ is odd for all $n \in \mathbb{N}$.

3 *(a) Give a recursive description of the number of ways of covering a $2 \times n$ chessboard with 2×1 tiles.*

If $n = 0$, there is only one way to tile a 2×0 chessboard – use no tiles. Similarly, if $n = 1$, there is only one way – use just one horizontal tile. Now suppose $n > 1$. We can either tile the $2 \times n$ chessboard by placing a horizontal tile across the first row, then tiling the remaining $2 \times (n-1)$ chessboard, or we can tile the first two rows with two vertical tiles, then tile the remaining $2 \times (n-2)$ chessboard.

(b) Deduce that the number of ways to tile a $2 \times n$ chessboard is the $(n+1)$th Fibonacci number.

Let T_n be the number of ways of tiling a $2 \times n$ chessboard. Then from (a), we see that for $n > 1$, $T_n = T_{n-1} + T_{n-2}$, since the number of ways of tiling the $2 \times n$ chessboard is the number of ways in which the first row is tiled with a horizontal tile, which is T_{n-1}, since all that remains is to tile the uncovered $2 \times (n-1)$ chessboard, plus the number of ways in which the first two rows are tiled with vertical tiles, which is T_{n-2}. Also, we know that $T_0 = F_{0+1}$, and $T_1 = F_{1+1}$, so we can show the result holds for all n by induction.

4 *Let A be the Ackermann function; find $A(3, 5)$. [Hint: start by finding recurrence relations for $A(1, n)$ and then $A(2, n)$ and solving them.]*

Note that $A(1,n) = A(0, A(1, n-1)) = A(1, n-1) + 1$, and $A(1,0) = 2$, so we can show by induction that $A(1,n) = n+2$ for all n – we've already checked the base case, and for the induction step, $n+2+1 = (n+1)+2$. Now we deduce that $A(2,n) = A(1, A(2, n-1)) = A(2, n-1) + 2$, and $A(2,0) = A(1,1) = 3$, so $A(2,n) = 2n+3$ – again, we've already checked the base case, while for the induction step, $2n+3+2 = 2(n+1)+3$.

We can now get:

$$
\begin{aligned}
&A(3,5) = A(2, A(3,4)) = 2A(3,4) + 3 = 2A(2, A(3,3)) + 3 = 2(2A(3,3) + 3) + 3 \\
&= 4A(2, A(3,2)) + 9 = 4(2A(3,2) + 3) + 9 = 8A(3,2) + 21 = 8A(2, A(3,1)) + 21 \\
&= 8(2A(3,1) + 3) + 21 = 16A(3,1) + 45 = 16A(2, A(3,0)) + 45 = 16(2A(3,0) + 3) + 45 \\
&= 32A(2,1) + 93 = 32 \times 5 + 93 = 253
\end{aligned}
$$

5 *Suppose $k = 2^n$. Let $a_1, a_2, \ldots, a_k$ be a set of k positive real numbers. Let M_n be the maximum value of $\frac{a_1 a_2 \cdots a_k}{(a_1 + a_2 + \cdots + a_k)^k}$, for any positive real numbers $a_1, \ldots, a_k$.*

(a)Show that M_n satisfies the recurrence $M_{n+1} \leqslant M_n^2 M_1^{(2^{n-1})}$. [Hint: rewrite the fraction $\frac{a_1 a_2 \cdots a_k}{(a_1 + \cdots + a^k)^k}$ as

$$
\frac{a_1 a_2 \cdots a_{2^{n-1}}}{(a_1 + \cdots + a_{2^{n-1}})^{2^{n-1}}} \times \frac{a_{2^{n-1}+1} \cdots a_k}{(a_{2^{n-1}+1} + \cdots + a_k)^{2^{n-1}}}
\times \frac{(a_1 + \cdots + a_{2^{n-1}})^{2^{n-1}} (a_{2^{n-1}+1} + \cdots + a_k)^{2^{n-1}}}{(a_1 + \cdots + a_k)^k}
$$

]

To maximise $\frac{a_1 a_2 \cdots a_k}{(a_1 + a_2 + \cdots + a_k)^k}$ when $k = 2^n$, rewrite the fraction as

$$
\frac{a_1 a_2 \cdots a_{2^{n-1}}}{(a_1 + \cdots + a_{2^{n-1}})^{2^{n-1}}} \times \frac{a_{2^{n-1}+1} \cdots a_k}{(a_{2^{n-1}+1} + \cdots + a_k)^{2^{n-1}}}
\times \frac{(a_1 + \cdots + a_{2^{n-1}})^{2^{n-1}} (a_{2^{n-1}+1} + \cdots + a_k)^{2^{n-1}}}{(a_1 + \cdots + a_k)^k}
$$

The first two fractions are both at most M_{n-1}, while the third fraction is at most $M_1^{2^{n-1}}$. Therefore, $M_n \leqslant M_{n-1}^2 M_1^{(2^{n-1})}$ as required.

(b) Find the values of M_n. [Hint: for the $n = 1$ case, note that $(a_1 - a_2)^2 > 0$.]

We note that $M_1 = \frac{1}{4}$, since $(a_1 - a_2)^2 > 0$, so $2a_1 a_2 \leqslant a_1^2 + a_2^2$, and so $4a_1 a_2 \leqslant (a_1 + a_2)^2$.

We now show that $M_n = \frac{1}{(2^n)^{(2^n)}}$ by induction on n. When $n = 0$, it is obvious. We have already checked $n = 1$. Now the recurrence relation becomes $M_n = \frac{M_{n-1}^2}{2^{(2^n)}}$. When we substitute $\frac{1}{(2^n)^{(2^n)}}$ for M_n, we get $\frac{1}{(2^n)^{(2^n)}} \leqslant \frac{1}{\left((2^{n-1})^{(2^{n-1})}\right)^2 2^{(2^n)}}$, which is true. This gives us that $M_n \leqslant \frac{1}{(2^n)^{(2^n)}}$. We need to show that this is an equality. To do this, we note that this value of M_n is attained when the a_n are all equal.

Bonus Question

6 *In the game Go, 2 players take turns to place stones of their colour (black or white) on the points in a* 19×19 *grid. A stone is captured if all of the neighbouring points (horizontally or vertically, but not diagonally) are occupied by stones of the opposite colour. On the other hand, if one of the neighbouring points is occupied by a stone of the same colour, then the two stones are considered like a single entity, i.e. they are both captured if all the neighbours of either stone (except for the ones on which the stones themselves are located) are occupied by stones of the opposite colour. Similarly for larger blocks of stones.*

Give a recursive definition of when a stone is captured (possibly as part of a larger group of captured stones).

The following is probably simplest:

A stone is captured if:

(i) All of its neighbouring points are occupied by stones of the opposite colour.

or

(ii) All of its neighbouring points are occupied, and all of its neighbours of the same colour would be captured if the stone were replaced by a stone of the opposite colour.