MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
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Homework Sheet 9
Model Solutions

Compulsory questions

1 *Let* $A = \{0, 1, 3, 5\}$*,* $B = \{x \in \mathbb{R} | 1 < x \leqslant 5\}$*, and* $C = \{x \in \mathbb{R} | x < 0.3 \vee x \geqslant 3\}$*. Find:*

(a) $A \cup B$

$A \cup B = \{x \in \mathbb{R} | (x = 0) \vee (1 \leqslant x \leqslant 5)\}$.

(b) $B \cap C$

$B \cap C = \{x \in \mathbb{R} | 3 \leqslant x \leqslant 5\}$.

(c) $A \setminus B$

$A \setminus B = \{0, 1\}$.

(d) $A \cup (B \cap C)$

$A \cup (B \cap C) = \{x \in \mathbb{R} | (x = 0) \vee (x = 1) \vee (3 \leqslant x \leqslant 5)\}$

(e) $(A \cup B) \cap C$

$(A \cup B) \cap C = \{x \in \mathbb{R} | (x = 0) \vee (3 \leqslant x \leqslant 5)\}$

(f) $P(A)$

$$P(A) = \{\emptyset, \{0\}, \{1\}, \{3\}, \{5\}, \{0, 1\}, \{0, 3\}, \{0, 5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{0, 1, 3\}, \{0, 1, 5\}, \{0, 3, 5\}, \{1, 3, 5\}, \{0, 1, 3, 5\}\}$$

2 *The* symmetric difference $A \triangle B$ *of two sets* A *and* B *is given by* $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

(a) Show that $(A \triangle B)^c = (A^c \cap B^c) \cup (A \cap B)$.

Expressing the symmetric difference in terms of complements, $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$, and so,

$$(A \triangle B)^c = (A \cap B^c)^c \cap (A^c \cap B)^c = (A^c \cup B) \cap (A \cup B^c)$$
$$= (A^c \cap A) \cup (A^c \cap B^c) \cup (B \cap A) \cup (B \cap B^c) = (A^c \cap B^c) \cup (A \cap B)$$

(b) Show that symmetric difference is associative, i.e. that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

$$(A \triangle B) \triangle C = ((A \cap B^c) \cup (A^c \cap B)) \cap C^c) \cup (((A^c \cap B^c) \cup (A \cap B)) \cap C)$$
$$= (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$$

Similarly,

$$A \triangle (B \triangle C) = A \cap ((B^c \cap C^c) \cup (B \cap C)) \cup (A^c \cap ((B^c \cap C) \cup (B \cap C^c))$$
$$= (A \cap B^c \cap C^c) \cup (A \cap B \cap C) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C)$$

So $(A \triangle B) \triangle C = A \triangle (B \triangle C)$

3 Let $A_1, A_2, \ldots$ *be an infinite collection of sets such that for any* n, $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$. *Can these* $A_1, A_2, \ldots$ *be chosen so that* $\bigcap_{i=1}^{\infty} A_i = \emptyset$*?*

This is possible: let $A_n = \left\{x \in \mathbb{R} \middle| 0 < x < \frac{1}{n}\right\}$, then $A_1 \cap A_2 \cap \cdots \cap A_n = A_n$, which is non-empty. However, $\bigcap_{i=0}^{\infty} A_i = \emptyset$.

4 Use the inclusion-exclusion principle to find the number of composite from 1 to 100 inclusive. Show your working. [Hint: Any number between 1 and 100 that is composite is divisible by one of 2,3,5 or 7 (Bonus question: Why?).]

Between 1 and 100, there are 50 multiples of 2, 33 multiples of 3, 20 multiples of 5, and 14 multiples of 7. There are 16 multiples of $3 \times 2 = 6$, 10 multiples of $5 \times 2 = 10$, 7 multiples of $7 \times 2 = 14$, 6 multiples of $3 \times 5 = 15$, 4 multiples of $3 \times 7 = 21$, and 2 multiples of $5 \times 7 = 35$. There are 3 multiples of $2 \times 3 \times 5 = 30$, 2 multiples of $2 \times 3 \times 7 = 42$, 1 multiple of $2 \times 5 \times 7 = 70$, and no multiples of $3 \times 5 \times 7$, or $2 \times 3 \times 5 \times 7$, so the total number of multiples of 2,3,5, or 7 between 1 and 100 inclusive is

$50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 = 78$. This includes the numbers 2,3,5, and 7, which are not composite, so the total number of composite numbers between 1 and 100 inclusive is 78-4=74.

The reason that any composite number between 1 and 100 is divisible by one of 2,3,5, or 7 is that if n is a composite number that is at most 100, then $n = ab$ for some a and b, both greater than 1. If a and b were both greater than 10, then $n = ab$ would be greater than 100, so by contradiction, one of a and b must be at most 10. Without loss of generality, suppose $a \leqslant 10$, then a has a prime factor (by unique prime factorisation) which must be one of 2, 3, 5, and 7, since these are the only prime numbers less than or equal to 10.