MATH 2600/STAT 2600, Theory of Interest
FALL 2013
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Midterm Examination
Model Solutions

1. *Mr. Almon receives an invoice for \$4,000, for payment within 50 days. He can get a 2% discount if he pays within the first 10 days. What is the largest rate of simple interest at which it would be worth his taking out a loan to get the discount.*

   His alternatives are to pay \$4,000 in 50 days, or to take out a loan to pay $4000\times 0.98 = \$3,920$ in 10 days. If he takes out the loan at simple interest rate i, he will have to repay $3920(1+\frac{40}{365}i)$ after 50 days. This worthwhile if this amount is less than \$4,000, which is

$$
\begin{aligned}
3920(1+\frac{40}{365}i) &\leqslant 4000\\
3920\frac{40}{365}i) &\leqslant 80\\
\\
i &\leqslant 18.6\%
\end{aligned}
$$

   So the largest rate is 18.6%.

2. *Dr. Baker buys a promissory note for \$6,000 in 150 days at 3% simple interest. After 80 days, she sells it to a bank, which discounts notes at 4% simple interest.*

   (a) How much does the bank pay for the note?

   The note pays $6000(1+\frac{150}{365}0.03) = 6073.97$ after 150 days. The amount the bank pays is given by $P(1+\frac{70}{365}0.04) = 6073.97$, or $P = 6027.73$.

   (b) What is Dr. Baker's rate of return?

   Dr. Baker invests \$6,000 and receives \$6027.73, 80 days later. This is a simple interest rate given by

$$
\begin{aligned}
6000(1+\frac{80}{365}i) &= 6027.73\\
6000\frac{80}{365}i &= 27.73\\
i &= 2.11\%
\end{aligned}
$$

3. *What rate of simple discount is equivalent to 5% simple interest over a period of 2 months?*

   We have $1-\frac{2}{12}d=\frac{1}{1+\frac{2}{12}0.05}$, which gives $d=6\left(1-\frac{1}{1+\frac{2}{12}0.05}\right)$, or $d=4.96\%$.

4. *Mr and Mrs. Carson are saving up for their children's education. They have two children, aged 10 and 12. They invest \$50,000 today at $j_{12}=5\%$ interest, and want to use this to pay \$40,000 for each child's education, when the child is 18 (at exactly this time of year). How much more money do they need to add to the fund in three years time, in order to have enough?*

   We set up an equation of value with focal date in 3 years' (36 months') time. This gives $50000(1+\frac{0.05}{12})^{36}+X=40000(1+\frac{0.05}{12})^{-36}+40000(1+\frac{0.05}{12})^{-60}$, or $X=40000(1+\frac{0.05}{12})^{-36}+40000(1+\frac{0.05}{12})^{-60}-50000(1+\frac{0.05}{12})^{36}=7533.65$.

5. *Which of the following interest rates is best for the lender?*

   (i) 9% compounded quarterly

   (ii) 9.2% compounded annually

   (iii) 8.9% compounded monthly

   (i) is an annual effective rate of $(1.0225)^4-1=9.31\%$. (iii) is an annual effective rate of $(1+\frac{0.089}{12})^{12}=9.27\%$, so (i) is best for the lender.

6. *What annual effective rate is equivalent to continuous compounding (constant force of interest) at 5%?*

   We have $1+i=e^c$, so $i=e^{0.05}-1=5.13\%$.

7. *If force of interest is given by $\delta_t=0.1+0.3t-0.1e^t$ over a one-year period, how much needs to be invested at the start of the period, to cover a bill for \$8,000 at the end of the year?*

   We have that the value at the end of the year of P invested at the start of the year is $Pe^{\int_0^1\delta_t dt}$, and we have $\int_0^1\delta_t dt=\int_0^1(0.1+0.3t-0.1e^t)dt=[0.1t+0.15t^2-0.1e^t]_0^1=0.35-0.1e=0.0781718$, so we have $P=8000e^{-0.0781718}=7398.44$.

8. *The stock of company ABC currently pays a dividend of \$0.30 every quarter. Every quarter the company increases the dividend by 1%. The current price for the stock (just after a dividend of \$0.30 is payed) is \$15. What interest rate is being used to value this stock?*

   At \$15, the real rate of interest per quarter is $\frac{0.3}{15}=0.02$. However, this is the real rate given by $\frac{i-0.01}{1.01}$, so we get $\frac{i-0.01}{1.01}=0.02$, so $i=3.02\%$ per quarter, or $j_4=12.08\%$.

9. *Mrs. Drake makes a loan of \$30,000 at $j_{12} = 7\%$. The loan is repaid over 6 years with equal monthly payments. When Mrs. Drake receives each payment, she immediately deposits it in an account which receives $j_{12} = 4\%$ interest. What yield does she make on her investment at the end of the 6 years?*

   The monthly payments are given by $Ra_{\overline{72}|\frac{0.07}{12}} = 30000$ or $R = \frac{30000\times\frac{0.07}{12}}{1-(1+\frac{0.07}{12})^{-72}} = 511.47$. These payments are deposited into an account which pays $j_{12} = 4\%$ interest, so the accumulated amount in this account is $511.47 s_{\overline{72}|\frac{0.04}{12}} = 511.47\frac{(1+\frac{0.04}{12})^{72}-1}{\frac{0.04}{12}} = 41542.92$. The annual yield on her investment is therefore given by $\left(\frac{41542.92}{30000}\right)^{\frac{1}{6}} - 1 = 5.58\%$. [or $j_{12} = 5.44\%$.]

10. *A company buys a machine for \$40,000. The machine is expected to last for 4 years, after which it will have a salvage value of \$8,000. Prepare a depreciation schedule using:*

    (a) The sum of digits method.

    (b) The constant percentage method

    We have $(1-d)^4 = \frac{8000}{40000} = \frac{1}{5}$, so $d = 1 - \left(\frac{1}{5}\right)^{\frac{1}{4}} = 0.33126$.

    (c) The straight line method

    (d) The compound interest method, with cost of capital $j_1 = 4\%$.

    $\frac{32000}{s_{\overline{4}|0.04}} = \frac{32000\times 0.04}{1.04^4-1} = 7535.68$.

	Sum of Digits		Constant Percentage		Straight Line		Compound Interest	
Year	Value	Depr.	Value	Depr.	Value	Depr.	Value	Depr.
1	40000	12800	40000	13250.39	40000	8000	40000	8476.62
2	27200	9600	26749.61	8861.07	32000	8000	31523.38	8150.59
3	17600	6400	17888.54	5925.75	24000	8000	23372.79	7837.11
4	11200	3200	11962.79	3962.79	16000	8000	15535.68	7535.68
5	8000		8000		8000		8000	

11. *A company are deciding between two machines. The first machine costs \$130,000, lasts for 8 years, after which it has a resale value of \$16,000, and has maintainance costs of \$4,000 every year. The second machine costs \$180,000, lasts for 9 years, with a resale value of \$22,000, and has fuel and maintainance costs of \$3,000 in the first year, and increasing by \$80 in each subsequent year. If the cost of capital is $j_1 = 8\%$, which machine has lower total capitalised cost?*

    The first machine has total cost of repurchases forming a perpetuity of \$114,000 every 8 years (interest rate $(1.08)^8 - 1$), which is a present value \$133,971.03. The maintainance costs form a perpetuity of \$4,000 every year, which has a present value of \$50,000, and the initial purchase is \$130,000, so the total capitalised cost is \$313,971.03.

The second machine has total cost of repurchases $158,000 every 9 years, so has present value 158,157.43. The maintainance has capitalised cost $\frac{3000}{0.08} = 37500$, and the initial purchase of $180,000 makes a total capitalised cost of $375,657.43, so the first machine has the lower total capitalised cost.

12. *Mr. Eccles takes out a loan for $8,000, to be repaid over 24 months at* $j_{12} = 8\%$. *Calculate the outstanding balance after 5 months using:*

    (a) *The retrospective form.*

    The regular payments are given by $Ra_{\overline{24}|\frac{0.08}{12}} = 8000$, so $R = \frac{8000\times\frac{0.08}{12}}{1-(1+\frac{0.08}{12})^{24}} = 361.82$.

    The value of the original loan is $8000(1+\frac{0.08}{12})^5 = 8270.25$. The value of the payments made is $361.82s_{\overline{5}|\frac{0.08}{12}} = 1833.37$, so the outstanding balance is $8270.25 - 1833.37 = 6436.87$.

    (b) *The prospective form.*

    The value of the remaining payments is $361.82a_{\overline{19}|\frac{0.08}{12}} = 361.82\frac{1-(1+\frac{0.08}{12})^{-19}}{\frac{0.08}{12}} = 6436.87$.

13. *A loan of $120,000 at* $j_1 = 8\%$ *is amortised with equal annual payments for 5 years.*

    (a) *Calculate the annual payments.*

    The annual payments are given by $Ra_{\overline{5}|0.08} = 120000$, so $R = \frac{120000\times 0.08}{1-1.08^{-5}} = 30054.77$.

    (b) *Draw up a complete amortisation schedule for the loan.*

Year	Outstanding Balance at Start	Interest	Payment	Principal Repaid
1	120000	9600	30054.77	20454.77
2	99545.23	7963.62	30054.77	22091.16
3	77454.06	6196.33	30054.77	23858.45
4	53595.62	4287.65	30054.77	25767.12
5	27828.49	2226.28	30054.77	27828.49

14. *Mrs. Finch takes out a 25-year mortgage for a loan of $200,000 at* $j_2 = 7\%$.

    (a) *Calculate the monthly payments required.*

    The monthly interest rate is $i = 1.035^{\frac{1}{6}} - 1 = 0.005750039$.

    The monthly repayments are given by $Ra_{\overline{300}|i} = 200000$, so $R = \frac{200000i}{1-(1+i)^{-300}} = 1400.84$ (rounded up).

    (b) *After 5 years, the interest rate rises to* $j_2 = 9\%$, *calculate the new monthly payments if she wishes to keep the mortgage over 25 years.*

    The value of the loan after 5 years is $200000(1.035)^{10} = 282119.75$. The value of the payments made is $1400.84s_{\overline{60}|i} = 1400.84\frac{1.035^{10}-1}{i} = 100031.17$, so the outstanding balance is $282119.75 - 100031.17 = 182088.59$.

The new monthly interest rate is $i = 1.045^{\frac{1}{6}} - 1 = 0.007363123$,

If she wishes to keep the mortgage over 25 years, then the new payments satisfy $Ra_{\overline{240}|i} = 182088.59$, or $R = \frac{182088.59i}{1-(1+i)^{-240}} = 1619.12$ (rounded up).

(c) If instead, she wishes to keep the mortgage payments the same, when will she finish paying off the mortgage?

If she wishes to keep the mortgage payments at 1400.84, then we have $1400.84a_{\overline{n}|i} = 182088.59$, so $1-(1+i)^{-n} = \frac{182088.59i}{1400.84}$, so $n = -\frac{\log\left(1-\frac{182088.59i}{1400.84}\right)}{1+i} =$ 429.22 months, or 35 years 9 months.

15. *Mr. and Mrs. Green buy a cottage, with a downpayment of \$50,000 and a 15-year mortgage for the remaining \$150,000 at $j_2 = 5\%$. There is a penalty of three times monthly interest on the outstanding balance for paying off the loan early. After 3 years, another company offers them a chance to refinance at $j_2 = 4.4\%$ for the remaining 12 years of the loan. Should they refinance?*

    The monthly interest is given by $i = 1.025^{\frac{1}{6}} - 1 = .004123915$.

    The monthly payments are given by $Ra_{\overline{180}|i} = 150000$, so $R = \frac{150000i}{1-(1+i)^{-180}} =$ 1182.19. After 3 years, the value of the original loan is $150000(1.025)^6 =$ 173954.01. The accumulated value of the payments made so far is $1182.92s_{\overline{36}|i} = 1182.19\frac{1.025^6-1}{i} = 45778.81$, so the outstanding balance is 128175.20. The penalty is $3 \times 128175.20 \times i = 1585.75$, so if they refinance, the outstanding balance is $128175.20+1585.75 = 129760.95$. If they refinance at $j_2 = 4.4\%$, which is a monthly rate of $i = 1.022^{\frac{1}{6}} - 1 = 0.003633501$, the new payments would satisfy $Ra_{\overline{144}|i} = 129760.95$, so $R = \frac{129760.95i}{1-1.022^{-24}} = 1158.93$ (rounded up), so they should refinance.

16. *Mrs. Horton buys a house in the US. She needs to borrow \$300,000 at $j_{12} = 7.2\%$, amortised over 15 years. There is also a financing fee of \$5,000. What is the APR for this loan?*

    The loan amount is \$305,000, so the monthly payments are given by $Ra_{\overline{180}|0.006} = 305000$ or $R = \frac{305000\times0.006}{1-1.006^{-180}} = 2775.64$. We want to find the monthly interest rate i, such that $2775.64a_{\overline{180}|i} = 300000$, or $a_{\overline{180}|i} = \frac{300000}{2775.64} = 108.0830$.

i	$a_{\overline{n}\|i}$
0.006	109.884
0.007	102.157
0.0062	108.271
0.0063	107.47
0.00622	108.11
0.00623	108.032

17. *A bank lends \$200,000 to Mr. and Mrs. Inglis. The loan is payed back with monthly interest-only payments at $j_{12} = 4\%$, with the principal returned*

as a lump sum after 15 years. After 8 years, the bank sells the loan to a private investor, who wishes to achieve an annual effective yield of 5.4%. How much does the investor pay for the loan?

We evaluate using Makeham's formula $K = 200000(1.054)^{-7} = 138403.07$. An annual effective rate of 5.4% is a monthly rate of $j = 1.045^{\frac{1}{12}} - 1 = 0.004392322$, so we have $P = 138403.07+(200000-138403.07)\frac{0.0033333333}{0.004392322} = 185148.98$.

18. *Mrs. Jeeves borrows \$6,000 for one year at 7% simple interest. After 3 months, she repays \$3,000.*

    If the loan is calculated using the merchant's rule, how much does she need to pay 8 months after the start of the loan, to pay off the debt?

    The original loan increases to $6000(1 + \frac{8}{12}0.07) = 6280$, while the first repayment grows to $3000(1 + \frac{5}{12}0.07) = 3087.50$, so the outstanding balance is $6280 - 3087.50 = 3192.50$.

19. *Dr. Kearns borrows \$600,000 at 6% simple interest for one year. The US rule is used to calculate the outstanding balance. After 4 months, he has \$60,000. He can earn simple interest at 3% on this money. When should he repay this money in order to minimise the outstanding balance at the end of the year?*

    If he repays the money after t years, amount repaid will be $60000(1+0.03t)$; the balance due at this time will be $600000(1+0.06(\frac{1}{3}+t))$, so the balance after the payment will be $600000(1 + 0.06(\frac{1}{3} + t)) - 60000(1 + 0.03t) = 552000 + 34200t$. The balance at the end of the year will therefore be $(552000 + 34200t)(1 + 0.06(\frac{2}{3} - t)) = 574080 + 2448t - 2050t^2$. This is minimised when $t = 0$, since t is at most $\frac{2}{3}$, so he should repay the money immediately.