MATH 3030, Abstract Algebra
Winter 2012
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Sample Final Examination

This practice exam deliberately has more questions than the real exam. Some of the theoretical questions are directly from the notes, and some are new, requiring a little thought. The questions from the notes are intended to provide a complete list of theorems from the last part of the course that you might be asked to prove. These questions deliberately focus on the part of the course after the midterm, because there are already a number of practice questions available on the material before the midterm.

Basic Questions

1. Which of the following pairs of numbers are conjugate over $\mathbb{Q}$?

   (a) $\sqrt{3}$ and $\sqrt{3}i$

   (b) $\sqrt{2}+\sqrt{3}$ and $\sqrt{2}-\sqrt{3}$

   (c) $\sqrt[5]{3}$ and $-\sqrt[5]{3}$.

2. Which of the following pairs of numbers are conjugate over $\mathbb{Q}(\sqrt{2})$?

   (a) $\sqrt[3]{3}$ and $\sqrt[3]{3}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

   (b) $\sqrt[4]{2}$ and $\sqrt[4]{2}i$

3. In $\mathbb{Q}(\sqrt[4]{3}, i)$, what is the fixed field of the automorphism σ which leaves $\mathbb{Q}$ fixed, and sends $\sqrt[4]{3}$ to $-\sqrt[4]{3}$ and sends i to $-i$.

4. Let α be a zero of x^4+3 in $\mathrm{GF}(5^4)$.

   (a) Let σ_5 be the Frobenius automorphism. Compute ${\sigma_5}^2(\alpha)$ [Give your answer in the basis $\{1, \alpha, \alpha^2, \alpha^3\}$.]

   (b) What is the fixed field of ${\sigma_5}^2$?

5. Find an element α such that $\mathbb{Q}(\sqrt{2+\sqrt{3}}, \sqrt{3+\sqrt{5}}) = \mathbb{Q}(\alpha)$ [Hint: to calculate differences between conjugates, try squaring the difference.]

6. Find a basis for the splitting field of x^4-3 over $\mathbb{Q}$.

7. Let f be an irreducible polynomial of degree 4 over a field F. Let K be the splitting field of f over F. Let the zeros of f be α, β, γ and δ. What is the orbit of $\alpha\beta\gamma+\delta$ under $G(K/F)$.

8. Write $\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{a}{d}+\frac{d}{a}+\frac{b}{c}+\frac{c}{b}+\frac{b}{d}+\frac{d}{b}+\frac{c}{d}+\frac{d}{c}$ as a rational function in the elementary symmetric functions $a+b+c+d$, $ab+ac+ad+bc+bd+cd$, $abc+abd+acd+bcd$ and $abcd$.

9. How many extension fields of $\mathbb{Q}$ are contained in the splitting field of $f(x) = x^4 - 7$?

10. (a) Is the regular 36-gon constructible?

    (b) Is the regular 60-gon constructible?

11. Find $\Phi_{26}(x)$ over $\mathbb{Q}$.

bonus Is $f(x) = x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 245$ solvable by radicals over $\mathbb{Q}$?

12. Is $f(x) = x^5 + x^3 + 2x + 3$ solvable by radicals over $\mathbb{Z}_7$?

13. Let α be a zero of $x^3 + 2x^2 + x + 1$ over $\mathbb{Z}_3$. What are the conjugates of α over $\mathbb{Z}_3$.

14. What are the conjugates of $\sqrt{\sqrt{2}+1}$ over $\mathbb{Q}$?

15. What is the degree of the splitting field of $x^3 + 2x^2 + 6x - 2$ over $\mathbb{Q}$?

Theoretical Questions

Results from Notes

16. Show that if α and β are elements of $\overline{F}$, show that there is an isomorphism $\sigma : F(\alpha) \longrightarrow F(\beta)$ such that $\sigma(a) = a$ for all $a \in F$ and $\sigma(\alpha) = \beta$, if and only if α and β are conjugate over F.

17. Show that the set of elements of a field E left fixed by a set S of automorphisms of E is a subfield of E.

18. Let E be a field, and let F be a subfield of E. Show

    (a) the set of automorphisms of E forms a group under function composition.

    (b) the subset of automorphisms of E that leave F fixed is a subgroup of this group.

19. Let F be a field of characteristic p. Show that the map $\sigma_p : F \longrightarrow F$ given by $\sigma_p(x) = x^p$ is an automorphism of F.

20. Let E be a finite extension of F, and let $\sigma : F \longrightarrow F'$ be an isomorphism. Show that there is an isomorphism $\hat{\sigma} : E \longrightarrow E'$, where E' is a subfield of $\overline{F'}$ and such that for all $a \in F$, we have $\hat{\sigma}(a) = \sigma(a)$.

21. Show that if $F \leqslant E \leqslant K$ then $\{K : F\} = \{K : E\}\{E : F\}$.

22. Show that an algebraic extension E is a splitting field over F if and only if every automorphism σ of $\overline{F}$ that leaves F fixed restricts to an automorphism of E — that is, for all $x \in E$, $\sigma(x) \in E$.

23. Show that if E is a splitting field of finite degree over F, then $|G(E/F)| = \{E : F\}$.

24. Let $f \in F[x]$ be irreducible. Show that all zeros of f have the same multiplicity.

25. Show that if E is a finite extension of F, and K is a finite extension of E, then K is separable over F if and only if K is separable over E and E is separable over F.

26. Show that a field of characteristic zero is perfect. [You may assume that if $g^n \in F[x]$, then $g \in F[x]$ whenever F has characteristic zero.]

27. Show that any finite field is perfect.

28. Show that if E is a finite separable extension of an infinite field F, then $E = F(\alpha)$ for some α in E.

29. Show that for any subgroup H of $G(K/F)$, where K is a finite normal extension of F, we have $\lambda(K_H) = H$.

30. Show that for any field $F \leqslant E \leqslant K$, where K is a finite normal extension of F, we have $K_{\lambda(E)} = E$.

31. Show that for any field $F \leqslant E \leqslant K$, where K is a finite normal extension of F, E is a normal extension of F if and only if $\lambda(E)$ is a normal subgroup of $G(K/F)$.

32. Show that a symmetric function in $y_1, \ldots, y_n$ over F is a rational function of the elementary symmetric functions.

33. Show that the Galois group of the nth cyclotomic extension of $\mathbb{Q}$ is isomorphic to the group of integers relatively prime to n under multiplication modulo n.

34. Show that a regular n-gon is constructible if and only if all odd prime divisors of n are Fermat primes, and n is not divisible by the square of any odd prime.

35. Let F be a field of characteristic zero. Show that if K is the splitting field of $x^n - a$ over F, then $G(K/F)$ is solvable.

36. Show that if E is a normal extension of F and K is an extension of F by radicals, with $F \leqslant E \leqslant K$, then $G(E/F)$ is solvable.

37. Show that any transitive subgroup of S_5 which contains a transposition is the whole of S_5.

38. Show that the quintic polynomial $x^5 - 3x + 6$ is not solvable by radicals over $\mathbb{Q}$.

39. Let $E = F(\alpha_1, \ldots, \alpha_n)$ be an algebraic extension of F. Show that any isomorphism σ from E to a subfield of $\overline{F}$, that leaves F fixed is uniquely determined by the values $\sigma(\alpha_1), \ldots, \sigma(\alpha_n)$.

New questions

40. Show that the algebraic closures of $\mathbb{Q}(\pi)$ and $\mathbb{Q}(e)$ are isomorphic.

41. Show that if $[E : F] = 2$, then E is a splitting field over F.

42. Let $E = F(\alpha)$ be a splitting field over F, and $[E : F] = 3$. Let the conjugates of α over F be β and γ. Suppose that $\sigma \in G(E/F)$ is such that $\sigma(\alpha) = \beta$. What is $\sigma(\beta)$?

43. Show that if α and β are both separable over F, then so is $\alpha + \beta$.

44. Is every algebraically closed field perfect? Give a proof or a counterexample.

45. Let $F \leqslant E \leqslant K$, where K is a normal extension of F. If $G(K/F)$ is abelian, show that $G(K/E)$ and $G(E/F)$ are both abelian.

46. Let $f = a_0 + a_1x + \cdots + a_nx^n$ be an irreducible polynomial in $F[x]$. Let α be a zero of F. Let K be the splitting field of f over F. Recall that the norm of α over F is given by

$$N_{K/F}(\alpha) = \Pi_{\sigma \in G(K/F)}\sigma(\alpha)$$

Suppose that $[K : F] = n$. Describe $N_{K/F}(\alpha)$ in terms of the coefficients of f.

47. Let $f(x) = x^3 + ax^2 + bx + c$ be an irreducible polynomial in $F[x]$, where F is a field of characteristic 3. Show that if f is not separable over F, then $a = b = 0$.

48. Let K be a finite normal extension of F. Let $\alpha \in K$. Show that $f(x) = \Pi_{\sigma \in G(K/F)}(x - \sigma(\alpha))$ is a power of $\mathrm{Irr}(\alpha, F)$.

49. Let m and n be coprime. Show that the mnth cyclotomic extension of $\mathbb{Q}$ is the splitting field of $\{x^m - 1, x^n - 1\}$ over $\mathbb{Q}$.

50. Show that $x^5 + 7x^2 - 7$ is not solvable by radicals over $\mathbb{Q}$.

51. Show that if K is a finite extension of F, and F is the fixed field of $G(K/F)$, then K is a splitting field over F.

52. Let f be an irreducible polynomial over F. Let α be a zero of f. Show that if α lies in a radical extension E of F, then so do all other zeros of f.

53. Let R be a radical extension of F, and let $F \leqslant E \leqslant R$ be an intermediate field. Must E be a radical extension of F? Give a proof or a counterexample.

Bonus Questions

54. If G is a group of automorphisms of E, and is isomorphic to S_3, must E be a splitting field over E_G?