MATH 3030, Abstract Algebra
FALL 2012
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Sample Midyear Examination

This sample exam is deliberately longer than the actually midyear. It also includes only questions from the topics covered after the midterm exam, although the midyear exam will include some questions on topics covered before then.

Basic Questions

1. Which of the following are rings: Justify your answers. [10 mins]

   (a) The collection of integers with the usual addition and multiplication given by $a * b = ab + a + b$.

   (b) The integers with the usual addition, and multiplication given by $a * b$ is the least common multiple of a and b.

   (c) The set of integers with the usual addition, and multiplication given by $a * b = 3ab$.

   (d) The collection of subsets of a set X with 5 elements, with addition given by symmetric difference and multiplication given by intersection. [The symmetric difference of two sets A and B is the set of elements that occur in exactly one of them.]

2. What are the units in the following rings: [15 mins]

   (a) $\mathbb{Z}_{22}$.

   (b) Numbers of the form $\frac{a+b\sqrt{2}i}{5}$ where a and b are integers.

3. Show that the set of numbers of the form $a + b\sqrt{5}$ where a and b are rational numbers is a field. [10 mins]

4. Which of the following rings are integral domains: [7 mins]

   (a) $\mathbb{Z}_{30}$.

   (b) The ring of 2×2 upper triangular matrices over $\mathbb{Z}$.

   (c) The collection of rational numbers where the denominator is a power of 2.

5. Factorise $x^4 + x^3 + 4x^2 + 24$:

   (a) over $\mathbb{Z}_5$. [5 mins]

   (b) over $\mathbb{Z}$. [6 mins]

6. Show that $f(x) = x^4 - x^3 + 3x^2 - 22x + 40$ is irreducible over $\mathbb{Z}$. [Hint: consider $x = y + 1$ and use Eisenstein's criterion.] [5 mins]

7. Find all solutions to the equation $x^2 - 3x + 8 = 0$ in $\mathbb{Z}_{12}$. [5 mins]

8. Find all prime numbers p such that $x-7$ is a factor of $x^4-4x^3+5x^2+4x-2$ in $\mathbb{Z}_p[x]$. [5 mins]

9. Find a generator for the multiplicative group of non-zero elements of $\mathbb{Z}_{29}$. [10 mins]

10. Show that $f(x) = x^3 - 2x^2 + 2$ is irreducible in $\mathbb{Z}_5$. [5 mins]

11. Find the remainder of 6^{1022} when divided by 11. [2 mins]

12. Find the remainder when 11^{123456} is divided by 21. [4 mins]

13. Find the remainder when $5^{5^{5^{5^{5^{5^{5^{5^{5}}}}}}}}$ is divided by 13. [6 mins]

14. Solve: [8 mins]

    (a) $8x \equiv 6 (mod 19)$

    (b) $2x \equiv 6 (mod 18)$

15. Describe the field of quotients of the integral domain $\{a + b\sqrt{2}i | a, b \in \mathbb{Z}\}$. [10 mins]

Theoretical Questions

16. (a) Show that the intersection of two subrings of a ring is a ring. [5 mins]

    (b) Show that the intersection of two subfields of a field is a subfield. [5 mins]

17. Show that the characteristic of an integral domain must be prime or 0. [5 mins]

18. Show that there is no field with exactly 6 elements. [10 mins]

19. Prove addition and multiplication are well-defined in the field of quotients of an integral domain. [7 mins]

20. Prove that a finite integral domain is a field. [10 mins]

21. Let D be an integral domain, and let F be a field of quotients of D. Let L be any field containing D. Prove that there is a homomorphism $\phi : F \longrightarrow L$ such that $\phi(x) = x$ for all $x \in L$. [10 mins]

22. State and prove the factor theorem for the polynomial ring over a field. [7 mins]

23. (a) Show that a polynomial of degree n in $F[x]$ for a field F can have at most n zeros. [7 mins]

(b) Deduce that the multiplicative group of non-zero elements in a finite field is cyclic. (Recall the classification theorem for finitely generated abelian groups.) [7 mins]