MATH 3030, Abstract Algebra
FALL 2012
Toby Kenney
Homework Sheet 1
Due: Friday 28th September: 3:30 PM

Basic Questions

1. Which of the binary operations in the following table are (a) Commutative (b) Associative?

(i)

	a	b	c	d
a	a	b	c	c
b	a	b	c	c
c	c	c	c	c
d	c	c	c	d

(ii)

	a	b	c
a	c	a	c
b	a	b	a
c	c	a	c

(iii)

	a	b	c
a	b	b	c
b	c	c	a
c	a	a	b

(iv)

	a	b	c
a	b	c	c
b	c	a	a
c	a	b	b

2. Which of the following produce well-defined binary operations? Justify your answers.

(a) $*$ on the complex numbers defined by $a*b$ is the solution to $x^2+ax+b = 0$.

(b) $*$ on the non-negative integers given by $a * b$ is the number of digits that a and b have in common (written in decimal).

(c) $*$ on the set of circles in the plane where $c_1 * c_2$ is the smallest circle which is tangent to both c_1 and c_2.

(d) $*$ on the set of intervals $[a, b]$ in the real numbers given by $[a, b]*[c, d] = \{xy|x \in [a, b], y \in [c, d]\}$.

3. Which pairs of the binary operations in Question 1 are isomorphic? Give an isomorphism if one exists and explain why one cannot exist if one does not exist.

4. Which of the following binary operations are groups? Justify your answers.

(i)

	a	b	c	d	e
a	a	b	c	d	e
b	b	d	a	e	c
c	c	e	d	b	a
d	d	a	e	c	b
e	e	c	b	a	d

(ii)

	a	b	c	d	e
a	a	b	c	d	e
b	b	a	e	c	d
c	c	d	a	e	b
d	d	e	b	a	c
e	e	c	d	b	a

(iii)

	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	a	e	f	c	d
c	c	f	a	e	d	b
d	d	e	f	a	b	c
e	e	d	b	c	f	a
f	f	c	d	b	a	e

(iv)

	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	d	c	d
d	d	d	d	d

(v)

	a	b	c
a	a	b	c
b	c	a	b
c	b	c	a

(iv)

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Theoretical Questions

5. Let $*$ and . be two binary operations on a set S, with the same identity element 1. Suppose further that $(a * b).(c * d) = (a.c) * (b.d)$ for any a, b, c and d in S. Prove that $*$ and . are the same operation, and further that this operation is commutative and associative. [Hint: consider $(a.1) * (1.b)$ and $(1.b) * (a.1)$.]

6. Show that the isomorphism relation between sets with a binary operation is an equivalence relation.

7. Let G be a finite group with identity e, and let x be an element of G. Show that there is some number n such that $x^n = e$.

8. For a fixed element a of a group G, show that the map ϕ given by $\phi(x) = axa^{-1}$ is an isomorphism from G to itself.

Bonus Questions

9. How many associative binary operations are there on a 3-element set?

10. Let S be a set with an associative binary operation $*$ such that for every element $x \in S$, there is a unique element x' such that $x * x' * x = x$. Prove that S is a group under $*$.