MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 4
Due: Friday 19th October: 3:30 PM

Basic Questions

1. In S_4, let H be the subgroup of permutations that fix 4. What is the left coset of H containing the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$?

2. Find the index of $\langle 4 \rangle$ in $\mathbb{Z}$.

3. Find the index of $\langle (0,2),(1,3) \rangle$ in $\mathbb{Z} \times \mathbb{Z}$.

4. Show that the group D_6 of symmetries of the regular hexagon is isomorphic to the direct product $S_3 \times \mathbb{Z}_2$.

5. (a) Show that a group of order 30 can have at most 2 subgroups of order 15. [Hint: the intersection of two subgroups is a subgroup. Use inclusion-exclusion principle to calculate the number of elements in the union of the subgroups.]

   (b) [bonus] Show that in fact a group of order 30 can have only one subgroup of order 15.

6. What is the order of $(3,7)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{21}$?

Theoretical Questions

7. For subgroups H and K of G, show that $(H : H \cap K) \leqslant (G : K)$.

8. Show that a group of even order must have an element of order 2.

9. Prove Theorem 10.14 that for subgroups $K \leqslant H \leqslant G$, if $(G : H)$ and $(H : K)$ are both finite, then $(G : K) = (G : H)(H : K)$.

10. Find a bijection (one-to-one and onto map) between the left cosets of H and the right cosets of H, and prove that it is a bijection.

11. Let H be a subgroup of G. Show that the set $N_G(H) = \{x \in G | xH = Hx\}$ is a subgroup of G.

12. Suppose G is a finite group, with subgroups H and K such that $|G| = |H||K|$, $H \cap K = \{e\}$ and $hk = kh$ for all $h \in H$ and $k \in K$. Show that G is isomorphic to $H \times K$.

13. If G, H and K are finitely generated abelian groups and $G \times K$ is isomorphic to $H \times K$, prove that G is isomorphic to H.

Bonus Questions

14. If G is a finitely generated abelian group, and H is a subgroup of G, must H also be a finitely generated abelian group? Give a proof or a counterexample.

15. (For students who know some Graph Theory) Hall's marriage theorem states:

    > Given a graph G whose vertices can be partitioned into two sets A and B of the same size, with all edges between one vertex in A and one vertex in B, it is possible to find a matching (a set of edges in the graph such that there is one edge at each vertex in A and one edge at each vertex in B) if and only if for any set A' of vertices in A the set of vertices in B adjacent to at least one vertex in A' has at least as many elements as A' and for any set B' of vertices in B the set of vertices in A adjacent to at least one vertex in B' has at least as many elements as B'.

    [Using this or otherwise] Show that: given a finite group G and a subgroup H, show that it is possible to choose a collection of elements of G with exactly one in every left coset of H and exactly one in every right coset of H.