MATH 3030, Abstract Algebra
FALL 2012
Toby Kenney
Homework Sheet 7
Due: Friday 16th November: 3:30 PM

Basic Questions

1. Which of the following are rings:

   (a) The collection of integers with the usual addition and multiplication given by $a * b = ab + a + b$.

   (b) The collection of positive rational numbers with multiplication and exponentiation. [That is $a + b = ab$ and $a \cdot b = a^b$.]

   (c) The set of real numbers which occur as solutions to quadratic equations with rational coefficients.

   (d) The set of integers with the usual addition, and multiplication given by $a * b = 3ab$.

2. What are the units in the following rings:

   (a) 2×2 matrices over $\mathbb{Z}$.

   (b) Numbers of the form $a + \frac{b}{\sqrt{2}}i$ where a and b are integers.

3. Show that the set of numbers of the form $a + b\sqrt{3}$ where a and b are rational numbers is a field.

4. Which of the following rings are integral domains:

   (a) $\mathbb{Z}_3 \times \mathbb{Z}_5$.

   (b) The ring of 2×2 upper triangular matrices over $\mathbb{Z}$.

   (c) The collection of rational numbers where the denominator is a power of 2.

5. Are the rings $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_{15}$ isomorphic?

6. Show that the only unital ring whose additive group is isomorphic to the integers, is the usual multiplication on the integers.

Theoretical Questions

7. A ring R is a Boolean ring if for any element $x \in R$, $x^2 = x$. Show that any Boolean ring is commutative.

8. (a) Show that the intersection of two subrings of a ring is a ring. (b) Show that the intersection of two subfields of a field is a subfield.

9. For a set X, let $P(X)$ denote the set of all subsets of X (this is called the power set of X). Show that $P(X)$ is a ring with the operations of symmetric difference and intersection.

10. Show that the characteristic of an integral domain must be prime or 0.

11. Show that there is no field with exactly 6 elements.

12. Show that the intersection of two subdomains of an integral domain is another subdomain.