MATH 3030, Abstract Algebra
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Homework Sheet 13
Model Solutions

Basic Questions

1. *Compute a composition series for S_4.*

   We know that A_4 is a normal subgroup of S_4, so we can choose A_4 as one element in the composition series. Next we need to find a normal subgroup of A_4. One possibility is the subgroup $K = \{e, (12)(34), (13)(24), (14)(23)\}$. We see that K is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ so we can pick any of the cyclic subgroups of order 2. Since A_4 has 12 elements, and K has 4, A_4/K must be isomorphic to $\mathbb{Z}_3$ (since that's the only group with 3 elements). Therefore we get the composition series:

   $$\{e\} \leqslant \{e, (12)(34)\} \leqslant K \leqslant A_4 \leqslant S_4$$

2. *Let $G = \mathbb{Z}_{30}$, let $K = \langle 6 \rangle$ and let $H = \langle 3 \rangle$. Give an explicit description of the isomorphism $G/H \longrightarrow (G/K)/(H/K)$.*

   We need to give the mapping from cosets of H to cosets of H/K in G/K. The cosets of H are represented by the elements 0, 1 and 2. The cosets of G/K are represented by 0, 1, 2, 3, 4 and 5, and the subgroup H/K contains just the cosets represented by 0 and 3. Therefore, $(G/K)/(H/K)$ is isomorphic to $\mathbb{Z}_6/\{0, 3\}$. The isomorphism of G/H to this sends the coset represented by 0, to the coset represented by 0, the coset represented by 1 to the coset represented by 1, and the coset represented by 2 to the coset represented by 2.

3. *In the group $G = S_4$, let $N = \{e, (12)(34), (13)(24), (14)(23)\}$, and let H be the subgroup of permutations that fix 1. Describe the isomorphism between $(HN)/N$ and $H/(H \cap N)$.*

   We see that $HN = S_4$, and that $H \cap N = \{e\}$, so we are describing an isomorphism from cosets of N to the subgroup of permutations that fix 1. The elements of H all represent different cosets of N, so in one direction, the isomorphism sends an element of H to the coset of N containing it. In the other direction, the isomorphism sends the coset σN to the unique element of σN which fixes 1. This can be obtained by multiplying σ by the permutation in N which sends 1 to $\sigma(1)$.

4. *Let $\phi : \mathbb{Z}_{15} \longrightarrow \mathbb{Z}_5$ be given by $\phi(1) = 3$. Let K be the kernel of ϕ. Explicitly describe the isomorphism given by the isomorphism theorem, between $\mathbb{Z}_{15}/K$ and $\mathbb{Z}_5$.*

The isomorphism sends the coset $m + K$ to the element $\phi(m) = 3m \pmod 5$ in $\mathbb{Z}_5$. In the other direction, it sends an element n of $\mathbb{Z}_5$ to the set of all elements in $\mathbb{Z}_{15}$ congruent to $2n$ modulo 5, which is a coset of K.

Theoretical Questions

5. *Let H and K be subgroups of G, with K normal in G, and such that $HK = G$ and $H \cap K = \{e\}$. Show that $G/K \cong H$.*

   By the isomorphism theorem, we know that $G/H = HK/K \cong H/(H \cap K) = H$.

6. *Show that the direct product of two solvable groups is solvable.*

   Let G and H be solvable groups. Let $\{e\} \leqslant G_1 \leqslant \cdots \leqslant G_n = G$, and $\{e\} \leqslant H_1 \leqslant \cdots \leqslant H_m = H$ be composition series for G and H. $G \times \{e\}$ is a normal subgroup of $G \times H$, and for two subgroups K and L of H, if K is a normal subgroup of L, then $G \times K$ is a normal subgroup of $G \times L$. Therefore

   $$\{e\} \leqslant G_1 \times \{e\} \leqslant \cdots \leqslant G \times \{e\} \leqslant G \times H_1 \leqslant \cdots \leqslant G \times H$$

   is a subnormal series for $G \times H$. (In fact it is a composition series). Furthermore, the quotient groups are all quotient groups from either the composition series of G, or the composition series of H, so they are abelian and simple, so this series is a composition series, and all the groups are abelian. Therefore, $G \times H$ is solvable.

7. *Show that a subgroup of a solvable group is solvable.*

   Let G be a solvable group, and let H be a subgroup of G. Let $\{e\} \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ be a composition series for G. Now consider $\{e\} \leqslant G_1 \cap H \leqslant \cdots \leqslant G_n \cap H = H$. We know that this is a subnormal series for H. By the second isomorphism theorem applied to the subgroup $(G_{i+1} \cap H)$ of G_{i+1}, and the normal subgroup G_i, we know that $(G_{i+1} \cap H)/(G_i \cap H) \cong ((G_{i+1} \cap H)G_i)/G_i$. Now $(G_{i+1} \cap H)G_i/G_i$ is a subgroup of G_{i+1}/G_i, which is simple (since the series is a composition series), and abelian, since G is solvable. This means it is cyclic of prime order, so that either $(G_{i+1} \cap H)/(G_i \cap H) \cong G_{i+1}/G_i$ or $(G_{\imath+1} \cap H)/(G_i \cap H)$ is the trivial group. Therefore, we see that, identifying equal elements in the series, we get a composition series for H. Furthermore, the quotients in this series are all quotients in the composition series for G. Therefore, they are all abelian, so H is solvable.

Bonus Questions

8. *Show that the homomorphic image of a solvable group is solvable.*

Let $f : G \longrightarrow H$ be an onto homomorphism, and let $\{e\} \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ be a composition series for G. Consider the series $\{e\} \leqslant f(G_1) \leqslant \cdots \leqslant f(G_n) = H$. We will show that after identifying equal terms, this is a composition series for H. We know that $f(G_i)$ is normal in $f(G_{i+1})$. We need to show that $f(G_{i+1})/f(G_i)$ is a quotient of G_{i+1}/G_i. However, there is a homomorphism $\phi : G_{i+1}/G_i \longrightarrow f(G_{i+1})/f(G_i)$ given by $\phi(xG_i) = f(x)f(G_i)$. This is clearly onto because for any coset $yf(G_i)$, we have that $y = f(x)$ for some x in G_{i+1}. Since G_{i+1}/G_i is simple, either $f(G_{i+1})/f(G_i)$ is isomorphic to G_{i+1}/G_i, or it is the trivial group. Therefore, the quotients in the composition series for H are all isomorphic to quotients in the composition series for G, and therefore abelian. Therefore, H is solvable.