MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 4
Due: Wednesday 17th October: 3:30 PM

Basic Questions

1. *In S_4, let H be the subgroup of permutations that fix 4. What is the left coset of H containing the permutation* $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$*?*

   This is the set of permutations σ such that $\sigma(4) = 3$, that is:

$$\left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}\right\}$$

2. *Find the index of $\langle 4\rangle$ in $\mathbb{Z}$.*

   The cosets of $\langle 4\rangle$ are $\{4n|n \in \mathbb{Z}\}$, $\{4n+1|n \in \mathbb{Z}\}$, $\{4n+2|n \in \mathbb{Z}\}$ and $\{4n+3|n \in \mathbb{Z}\}$, so the index is 4.

3. *Find the index of $\langle (0,2),(1,3)\rangle$ in $\mathbb{Z}\times\mathbb{Z}$.*

   The cosets in question are represented by $(0,0)$,$(0,1)$,$(1,0)$,$(1,1)$,$(2,0)$ and $(2,1)$, so the index is 6.

4. *Show that the group D_6 of symmetries of the regular hexagon is isomorphic to the direct product $S_3 \times \mathbb{Z}_2$.*

   S_3 is the group of symmetries of an equilateral triangle. We can choose three alternate vertices around the hexagon, and consider the symmetries which fix those three vertices. All symmetries of this triangle are symmetries of the whole hexagon, so we have expressed S_3 as a subgroup of D_6. We need to find an element of order 2 which commutes with all elements in this subgroup. We see that rotation by $180°$ about the centre of the hexagon is such an element.

5. *(a) Show that a group of order 30 can have at most 2 subgroups of order 15. [Hint: the intersection of two subgroups is a subgroup. Use inclusion-exclusion principle to calculate the number of elements in the union of the subgroups.]*

   Let G have order 30. Let H_1, H_2 and H_3 be distinct subgroups of order 15. The intersections $H_1 \cap H_2$, $H_1 \cap H_3$ and $H_2 \cap H_3$ are subgroups of

H_1 or H_2, so their order must divide 15. Therefore, the largest order they can possibly have is 5. Also, since the identity is contained in all three subgroups, we must have that Therefore, we have that $|H_1 \cup H_2 \cup H_3| \geqslant |H_1| + |H_2| + |H_3| - |H_1 \cap H_2| - |H_2 \cap H_3| - |H_2 \cap H_3| + |H_1 \cap H_2 \cap H_3| \geqslant 15+15+15-5-5-5+1 = 31$, which is impossible since $|H_1 \cup H_2 \cup H_3| \subseteq G$.

(b) [bonus] Show that in fact a group of order 30 can have only one subgroup of order 15.

Suppose that H_1 and H_2 are two distinct subgroups of G, with $|G| = 30$ and $|H_1| = |H_2| = 15$. We know that $|H_1 \cap H_2| \leqslant 5$, so $(H_1 : H_1 \cap H_2) \geqslant 3$. Let x, y and z be representatives of three different left cosets of $H_1 \cap H_2$ in H_1. That is, none of $x^{-1}y$, $x^{-1}z$ or $y^{-1}z$ is in $H_2 \cap H_1$. However, since x, y and z are all in H_1, and H_1 is a subgroup, none of $x^{-1}y$, $x^{-1}z$ or $y^{-1}z$ is in H_2. That is, x, y and z must be in different cosets of H_2 in G. However, $(G : H_2) = \frac{30}{15} = 2$, so this is impossible. Therefore, there can be at most one subgroup of G of order 15.

6. *What is the order of $(3, 7)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{21}$?*

   3 has order 2 in $\mathbb{Z}_6$, and 7 has order 3 in $\mathbb{Z}_{21}$, so $(3, 7)$ has order $3 \times 2 = 6$ in $\mathbb{Z}_6 \times \mathbb{Z}_{21}$.

Standard Questions

7. *For subgroups H and K of G, show that $(H : H \cap K) \leqslant (G : K)$.*

   Pick a set L containing one representative of every left coset of $H \cap K$ in H. Now for any two element x and y of L, x and y are elements of H, so $x^{-1}y \in H$, but they are in different left cosets of $H \cap K$, so $x^{-1}y \notin H \cap K$. However, this means $x^{-1}y \notin K$, so x and y are in different cosets of K in G, which proves that $(H : H \cap K) \leqslant (G : K)$.

8. *Show that a group of even order must have an element of order 2.*

   If x has order $2n$ for some n, then x^n has order 2, so we just need to show that the group must have an element of even order. G is the union of all its cyclic subgroups, and the intersection of any two cyclic subgroups is another cyclic subgroup, so suppose that the cyclic subgroups of G are $H_1, \ldots, H_k$ and that all of them have odd order. Let K_i be the set of non-identity elements of H_i. If all elements of G have odd order, the intersection of any set of the K_i must have odd order, and the set of non-identity elements of G is the union of all the K_i. Now by the inclusion-exclusion principle, we have that $|G| - 1 = |K_1| + \cdots + |K_n| - |K_1 \cap K_2| - \cdots - |K_{k-1} \cap K_k| + \cdots + (-1)^k |K_1 \cap \cdots \cap K_k|$. However, all the terms in this sum are even, so $|G| - 1$ must be even, and therefore, $|G|$ must be odd.

9. *Prove Theorem 10.14 that for subgroups $K \leqslant H \leqslant G$, if $(G : H)$ and $(H : K)$ are both finite, then $(G : K) = (G : H)(H : K)$.*

   Let $x_1, \ldots, x_n$ be a set of coset representatives of H in G, and let $y_1, \ldots, y_m$ be a set of coset representatives of K in H. We will show that $\{x_i y_j | 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ is a set of left coset representatives of K in G. First we show that any two of these elements are in different cosets of K. That is, if $(x_i y_j)^{-1} x_k y_l \in K$ then $i = k$ and $j = l$. We note that if $(x_i y_j)^{-1} x_k y_l = {y_j}^{-1} {x_i}^{-1} x_k y_l \in H$, then since y_j and y_l are both in H, we must have ${x_i}^{-1} x_k \in H$, so since the set $\{x_1, \ldots, x_n\}$ is a set of coset representatives, we must have $i = k$. Now we have that ${y_j}^{-1} x_i^{-1} x_i y_l = {y_j}^{-1} y_l \in K$, which gives $j = l$ as required.

   Finally, we need to show that any element of G is in the same coset as one of the products $x_i y_j$. Now any element $z \in G$ must be in the same coset of H as some x_i. That is, for some i, we have ${x_i}^{-1} z \in H$. Now since ${x_i}^{-1} z \in H$, it must be in one of the cosets of K in H. That is, it must have ${y_j}^{-1}({x_i}^{-1} z) \in K$ for some j. However, we now have ${y_j}^{-1}({x_i}^{-1} z) = ({y_j}^{-1} {x_i}^{-1}) z = (x_i y_j)^{-1} z \in K$ for some i and j. That is, z is in the same coset as some $x_i y_j$ as required.

10. *Find a bijection (one-to-one and onto map) between the left cosets of H and the right cosets of H, and prove that it is a bijection.*

    One bijection is given by $f(X) = \{x^{-1} | x \in X\}$. We need to show that f sends left cosets to right cosets, and that it is a bijection. We will show that f also sends right cosets to left cosets, and it is clear that $f(f(X)) = X$, so that f must be a bijection between left and right cosets.

    If X is a left coset of H, then $\{x^{-1} | x \in X\}$ is a right coset, since if $x, y \in X$, then $x^{-1}(y^{-1})^{-1} = x^{-1} y = (y^{-1} x)^{-1} \in H$, so that x^{-1} and y^{-1} are in the same right coset of H. On the other hand, suppose z is in the same right coset of H as x^{-1}. Now $z(x^{-1})^{-1} = zx = (x^{-1} z^{-1})^{-1} \in H$. Since H is a subgroup, we deduce that $x^{-1} z^{-1} \in H$. This means that z^{-1} is in the same left coset of H as x as required.

11. *Let H be a subgroup of G. Show that the set $N_G(H) = \{x \in G | xH = Hx\}$ is a subgroup of G.*

    Let $x, y \in N_G(H)$. Now $xyH = x(yH) = x(Hy) = (xH)y = Hxy$, so $xy \in N_G(H)$. Also $x^{-1} H = \{x^{-1} h | h \in H\} = \{x^{-1} h^{-1} | h \in H\} = \{x^{-1} h x x^{-1} | h \in H\} = \{x^{-1} k x^{-1} | k \in Hx\} = \{x^{-1} k x^{-1} | k \in xH\} = \{x^{-1} x h x^{-1} | h \in H\} = \{h x^{-1} | h \in H\} = Hx^{-1}$, so $x^{-1} \in N_G(H)$, so $N_G(H)$ is a subgroup.

12. *Suppose G is a finite group, with subgroups H and K such that $|G| = |H||K|$, $H \cap K = \{e\}$ and $hk = kh$ for all $h \in H$ and $k \in K$. Show that G is isomorphic to $H \times K$.*

    Consider the set of elements $\{hk | h \in H, k \in K\}$ of G. Suppose two of these elements are equal, i.e. $hk = h'k'$ for elements $h, h' \in H$ and

$k, k' \in K$. We get $kk'^{-1} = h^{-1}hkk'^{-1} = h^{-1}h'k'k'^{-1} = h^{-1}h'$, but $kk'^{-1} \in K$ and $h^{-1}h' \in H$, so this product must be in $H \cap K = \{e\}$. That is, we must have $kk'^{-1} = h^{-1}h' = e$, or $k = k'$ and $h = h'$. Therefore, we get that $|\{hk|h \in H, k \in K\} = |H||K| = |G|$, so $G = \{hk|h \in H, k \in K\}$.

We now form the isomorphism $G \xrightarrow{\phi} H \times K$ by $hk \mapsto (h, k)$. By the preceding argument, this is well-defined and a bijection. Furthermore, it is a homomorphism because $\phi(hk)\phi(h'k') = (h, k)(h', k') = (hh', kk')$, while $\phi((hk)(h'k')) = \phi(h(kh')k') = \phi(h(h'k)k') = \phi(hh'kk') = (hh', kk')$.

13. *If G, H and K are finitely generated abelian groups and $G \times K$ is isomorphic to $H \times K$, prove that G is isomorphic to H.*

By the structure theorem for finitely generated abelian groups, we have that $G \cong \mathbb{Z}_{i_1} \times \cdots \times \mathbb{Z}_{i_n} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, $H \cong \mathbb{Z}_{j_1} \times \cdots \times \mathbb{Z}_{j_m} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ and $K \cong \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_l} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, where each i_a, j_b and k_c is a prime power. Furthermore, these descriptions are unique, and the descriptions of $G \times H$ and $G \times K$ as these products must therefore be the same. That is $G \times K \cong \mathbb{Z}_{i_1} \times \cdots \times \mathbb{Z}_{i_n} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_l} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ and $H \times K \cong \mathbb{Z}_{j_1} \times \cdots \times \mathbb{Z}_{j_m} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_l} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$. Since these are isomorphic, we must have that $\{i_1, \ldots, i_n\} = \{j_1, \ldots, j_m\}$, and the number of copies of $\mathbb{Z}$ must be the same. This forces $G \cong H$.

Bonus Questions

14. *If G is a finitely generated abelian group, and H is a subgroup of G, must H also be a finitely generated abelian group? Give a proof or a counterexample.*

15. *(For students who know some Graph Theory) Hall's marriage theorem states*

> *Given a graph G whose vertices can be partitioned into two sets A and B of the same size, with all edges between one vertex in A and one vertex in B, it is possible to find a matching (a set of edges in the graph such that there is one edge at each vertex in A and one edge at each vertex in B) if and only if for any set A' of vertices in A the set of vertices in B adjacent to at least one vertex in A' has at least as many elements as A' and for any set B' of vertices in B the set of vertices in A adjacent to at least one vertex in B' has at least as many elements as B'.*

[Using this or otherwise] Show that: given a finite group G and a subgroup H, show that it is possible to choose a collection of elements of G with exactly one in every left coset of H and exactly one in every right coset of H.

For the group G, form a bipartite graph whose vertices are the left cosets of H and the right cosets of H, and with an edge from aH to Ha for any $a \in G$. Consider a set of left cosets of H. The neighbours of this set are the right cosets containing elements of the union of these left cosets. If the number of left cosets is n, then the total number of elements in the union is $n|H|$. This many elements cannot be contained in fewer than n right cosets of H. Therefore there are at least as many neighbours as in the set. The same argument works for sets of right cosets. Therefore, the conditions for Hall's marriage theorem hold, and we have a matching where every left coset is paired with a right coset that has an element in common with it. Selecting one of these elements for each left coset gives a set of elements with exactly one from every left coset and exactly one from every right coset.