MATH 3030, Abstract Algebra
FALL 2012
Toby Kenney
Homework Sheet 5
Due: Wednesday 24th October: 3:30 PM

Basic Questions

1. *Which of the following functions are homomorphisms?*

(a) $f : S_5 \longrightarrow S_3$ sending ϕ to the permutation obtained by restricting ϕ to $\{1,2,3\}$ and then relabelling the image of $\{1,2,3\}$ as $\{1,2,3\}$ in order. For example, if $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$, then the image of $\{1,2,3\}$ is $\{2,4,5\}$, so we relabel in order $2 \mapsto 1$, $4 \mapsto 2$ and $5 \mapsto 3$. This gives $f(\phi) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

This is not a homomorphism. For example $f\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$ and $f\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix}$ are both the identity, but $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$, and $f\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

(b) $f(x) = \pi x$ from $\mathbb{R}$ to itself (with $+$ as the group operation).

This is a homomorphism, since $f(x+y) = \pi(x+y) = \pi x + \pi y = f(x) + f(y)$.

(c) Let G be the group of 2×2 upper triangular real matrices with non-zero diagonal entries. Let $f : G \longrightarrow \mathbb{R}^*$ be the function sending a matrix in G to its bottom-right element.

This is a homomorphism, since $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & bc' + ab' \\ 0 & cc' \end{pmatrix}$.

(d) $f(x) = e^x$ from $\mathbb{R}$ with $+$ as the group operation to $\mathbb{R}^*$ with multiplication as the group operation.

This is a homomorphism, since $e^{x+y} = e^x e^y$.

2. *Which of the following subgroups are normal?*

(a) The rational numbers as a subgroup of the real numbers.

This is normal, since the real numbers are an abelian group, so all subgroups are normal.

(b) The subgroup of S_4 generated by $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$.

This is not normal. The left and right cosets containing $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ are $\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \right\}$ respectively.

(c) The group of complex $n \times n$ matrices X for which $\det(X)^{34} = 1$, as a subgroup of the group of complex $n \times n$ matrices with non-zero determinant. [Hint: recall from linear algebra that $\det(AB) = \det(A)\det(B)$.]

Since $\det(AB) = \det(A)\det(B)$, the determinant function is a homomorphism from the group of complex $n \times n$ matrices with non-zero determinant to the complex numbers, and the function $f(x) = x^{34}$ is a homomorphism from the complex numbers to itself. Therefore $g(X) = \det(X)^{34}$ is a homomorphism, so its kernel is a normal subgroup.

3. *Find the kernel and image of the following homomorphisms.*

   (a) Let G be the the group of symmetries of a cube. Define $f : G \longrightarrow S_4$ by the induced permutation on the diagonals.

   The kernel is the group of symmetries of the cube which fix all the diagonals. The only two symmetries which do this are the identity and the symmetry which sends each vertex of the cube to the opposite vertex.

   The image is the whole of S_4, since any permutation of the diagonals can be achieved by a symmetry of the cube. This is easy to see because any transposition can be achieved, and transpositions generate S_4.

   (b)

4. *Show that the function $G \xrightarrow{f} G$ given by $f(x) = x^2$ on a group G is a homomorphism if and only if G is abelian.*

   If G is abelian, then we have $f(xy) = (xy)^2 = xyxy = xxyy = x^2y^2 = f(x)f(y)$, so f is a homomorphism. Conversely, if f is a homomorphism, we get $xyxy = f(xy) = f(x)f(y) = x^2y^2$, so $x^{-1}xyxyy^{-1} = x^{-1}x^2y^2y^{-1}$, i.e. $xy = yx$, for any two elements x and y, so G is abelian.

Standard Questions

5. *Show that the composite of two homomorphisms is a homomorphism.*

   Let $G \xrightarrow{f} H$ and $H \xrightarrow{g} K$ be homomorphisms. We want to show that $G \xrightarrow{gf} K$ is also a homomorphism. We have that $gf(xy) = g(f(xy)) = g(f(x)(f(y)) = g(f(x))g(f(y)) = gf(x)gf(y)$ as required.

6. *Show that a homomorphism of groups $G \xrightarrow{\phi} G'$ is an isomorphism if and only if there is a homomorphism $G' \xrightarrow{\phi'} G'$ such that the composites $\phi\phi'$ and $\phi'\phi$ are both the identity homomorphism.*

   Suppose ϕ is an isomorphism. Then it is a bijection, so ϕ^{-1} is a function. We need to show that it is a homomorphism. However, we know that $\phi(\phi^{-1}(x)\phi^{-1}(y)) = \phi(\phi^{-1}(x))\phi(\phi^{-1}(y)) = xy$. This gives that $\phi^{-1}(xy) = \phi^{-1}(x)\phi^{-1}(y)$, as required.

   Conversely, suppose that there is a homomorphism ϕ' as described. Then as a function ϕ' must be the inverse of ϕ, which must therefore be a bijection.

7. *Let $\sim$ be an equivalence relation on a group G such that whenever $x \sim x'$ and $y \sim y'$, we also have $xx' \sim yy'$.*

   (a) Show that the subset $\{x \in G | x \sim e\}$, where e is the identity element of G, is a normal subgroup H.

   Suppose we have $x \sim e$, and $y \sim e$. Then we have $xy \sim ee = e$. Also, since $\sim$ is an equivalence relation, we have $x^{-1} \sim x^{-1}$, and thus $e = x^{-1}x \sim x^{-1}e = x^{-1}$, and by symmetry, $x^{-1} \sim e$. We have therefore shown that H is a subgroup of G. We need to show that it is normal. That is, we need to show that for any $a \in G$, $Ha = aH$. We will show that both of these sets are $\{x \in G | x \sim a\}$. If $x \in H$, then we have $x \sim e$, and since $a \sim a$, this gives $ax \sim ae = a$ and $xa \sim ea = a$. Conversely, suppose $x \sim a$, then since $a^{-1} \sim a^{-1}$, we get $a^{-1}x \sim a^{-1}a = e$, so $x = aa^{-1}x \in aH$, and similarly, $x = xa^{-1}a \in Ha$.

   (b) Show that the equivalence relation $\sim$ is given by $x \sim y$ if and only if $xy^{-1} \in H$.

   If $x \sim y$, then since $y^{-1} \sim y^{-1}$, we get $xy^{-1} \sim yy^{-1} = e$, i.e. $xy^{-1} \in H$. Conversely, if $xy^{-1} \in H$, then since $xy^{-1} \sim e$ and $y \sim y$, we get $x = xy^{-1}y \sim ey = y$.

8. *Show that any subgroup of index 2 is normal.*

   Let $H \leqslant G$ with $(G : H) = 2$. Since G is the disjoint union of the two cosets of H, one coset must be H, and the other must be its complement $G \setminus H$. This applies to both right and left cosets, so the right and left cosets must be equal. Therefore, H is normal in G.

9. *Show that if $H \leqslant G$ and N is a normal subgroup of G, then $N \cap H$ is a normal subgroup of H.*

   Let $a \in H$. We want to show that for any $x \in N \cap H$, we have $axa^{-1} \in N \cap H$. However, since N is a normal subgroup of G, we have that $axa^{-1} \in N$, and since a and x are both in H, we have that $axa^{-1} \in H$. Therefore, $axa^{-1} \in N \cap H$ as required.

10. *(a) Show that the intersection of two normal subgroups is another normal subgroup.*

Let M and N be normal subgroups of G. We want to show that for $x \in M \cap N$ and $a \in G$, we have $axa^{-1} \in M \cap N$. However, since M is normal, we have $axa^{-1} \in M$, and similarly, $axa^{-1} \in N$, so we get $axa^{-1} \in M \cap N$.

(b) Show that the subgroup generated by two normal subgroups is normal.

Let M and N be normal subgroups of G, and let $H = \langle M, N \rangle$. We want to show that for $x \in H$ and $a \in G$, we have $axa^{-1} \in H$. Now x is a product $x_1 \cdots x_n$, where each x_i is either in M or N. Now $axa^{-1} = ax_1a^{-1}ax_2a^{-1} \cdots ax_na^{-1}$, and since M and N are normal, every ax_ia^{-1} is either in M or N, so that $axa^{-1} \in \langle M, N \rangle$.

Bonus Questions