MATH 3030, Abstract Algebra
FALL 2012
Toby Kenney
Homework Sheet 6
Due: Friday 2nd November: 3:30 PM

Basic Questions

1. *Show that A_5 is simple. [Hint: consider the conjugacy classes].*

   The conjugacy classes of A_5 have sizes 1, 20, 15, 12 and 12. A normal subgroup is a union of conjugacy classes, so its size must be made up of a sum of these numbers. However, it must also be a subgroup, so its size must divide 60. Furthermore, this subgroup must contain the identity (the conjugacy class of size 1). If the size is odd, then it must be at most 15, and this is clearly impossible. Therefore, a normal subgroup must have even order, and must therefore contain the conjugacy class of size 15. Its size must therefore be either 20 or 30, since it must be at least 16. However, it is easy to check that no union of conjugacy classes including the identity has size 20 or 30. Therefore, A_5 has no non-trivial normal subgroups.

2. *(a) Calculate the commutator subgroup of D_6.*

   If a and b are two reflections, then they are self inverse, and ab is a rotation, so that $aba^{-1}b^{-1}$ is the square of a rotation — that is, it is a rotation by a multiple of $120°$. If a is a reflection, and b is a rotation, then $aba = b^{-1}$, so that $abab^{-1} = b^{-2}$ is also a rotation by a multiple of $120°$. Finally, if a and b are both rotations, then they commute, so the commutator is the identity. That is, the commutator subgroup consists of the subgroup of rotations by multiples of $120°$.

   (b) Calculate the factor group of D_6 over its commutator subgroup.

   The factor group has 4 elements, since the commutator subgroup has 3 elements. Furthermore, of the elements not in the commutator subgroup, 2 have order 6, and the remainder have order 2. Therefore, the subgroup cannot be $\mathbb{Z}_4$, since that would require an element with order divisible by 4. Therefore, the subgroup must be $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3. *Calculate the centre of D_8.*

   It is easy to see that two reflections do not commute, unless the angle between them is $90°$, so that no reflections are in the centre of D_8. Furthermore, if a is a reflection, and b is a rotation, then we have $ab = b^{-1}a$, so that a rotation is in the centre if and only if it is self-inverse. This happens only for a rotation by $180°$. Therefore, the centre of D_8 consists of just the identity and rotation by $180°$.

4. *If H is an abelian normal subgroup of G, must H be contained in $Z(G)$? Give a proof or a counterexample.*

   H does not need to be contained in the centre of G. The simplest example is S_3, which has trivial centre, but has A_3 as an abelian normal subgroup isomorphic to $\mathbb{Z}_3$.

Standard Questions

5. *Show that if G is a simple group, and $G \xrightarrow{\phi} H$ is a homomorphism of G onto H, then either H is trivial, or ϕ is an isomorphism.*

   The kernel of ϕ is a normal subgroup of G, so it must either be the trivial subgroup, or the whole of G. If it is the whole of G, then H must be trivial. On the other hand, if the kernel of ϕ is trivial, then ϕ is one-to-one, and since it is also onto, then it must be an isomorphism.

6. *Let $G \xrightarrow{\phi} A$ be a homomorphism from G to an abelian group, and let C be the commutator subgroup of G. Show that there is a homomorphism $G/C \xrightarrow{\phi'} A$, such that ϕ is the composite $G \longrightarrow G/C \xrightarrow{\phi'} A$.*

   This is equivalent to showing that C is contained in the kernel of ϕ. However, for any $x, y \in G$, we have that $\phi(xyx^{-1}y^{-1}) = \phi(x)\phi(y)\phi(x^{-1})\phi(y^{-1}) = \phi(x)\phi(x^{-1})\phi(y)\phi(y^{-1}) = e$, since A is abelian. Therefore, C is a subgroup of the kernel of ϕ.

7. *Show that if $G/Z(G)$ is cyclic, then G is abelian.*

   Let $xZ(G)$ be a generator of $G/Z(G)$. Since $G/Z(G)$ is cyclic, the set $\{x^n|n \in \mathbb{Z}\}$ of powers of x contains a representative of every coset of $Z(G)$. We want to show that G is abelian. Let a and b be two elements of G. We know that $a \in x^iZ(G)$ and $b \in x^jZ(G)$ for some $i, j \in \mathbb{Z}$. Suppose that $a = x^ic$ and $b = x^jd$ where $c, d \in Z(G)$. Now we have that $ab = x^icx^jd = x^ix^jcd = x^{i+j}dc = x^jdx^ic = ba$, so G is abelian.

8. *Show that the group of inner automorphisms of a group G is a normal subgroup of the group of all automorphisms of G.*

   Let $a \in G$, and let ϕ be an automorphism of G. We need to show that if θ_a is the inner automorphism of conjugation by a, then $\phi\theta_a\phi^{-1}$ is an inner automorphism of G. In fact, we will show that it is the inner automorphism corresponding to $\phi(a)$. This is because $\phi\theta_a\phi^{-1}(x) = \phi(a\phi^{-1}(x)a^{-1}) = \phi(a)\phi(\phi^{-1}(x))\phi(a^{-1}) = \phi(a)x(\phi(a))^{-1}$ as required.

9. *Let $N \leqslant H \leqslant G$, and N be a normal subgroup of G.*

   (a) Show that N is a normal subgroup of H.

   This is clear, since any $a \in H$ is also a member of G, so $aHa^{-1} = H$.

(b) Suppose H/N is a normal subgroup of G/N. Show that H is a normal subgroup of G.

Let $g \in G$. We want to show that for any $h \in H$, $ghg^{-1} \in H$. Since H/N is normal in G/N, we have that $ghg^{-1}N \subseteq H$, so in particular, $ghg^{-1} \in H$.

Bonus Questions

10. *Give an example of a group whose commutator subgroup is non-trivial and abelian.*