MATH/STAT 3360, Probability
FALL 2013
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Sample Final Examination

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. X is normally distributed with mean 2 and standard deviation 3. Y is normally distributed with mean 4 and standard deviation 4. What is $P(Y > X)$?

   (a) If X and Y are independent? [5 mins]

   (b) If X, Y have a multivariate normal distribution, and $\text{Cov}(X, Y) = 2$? [10 mins]

2. (R, Θ) have joint density function $f_{(R,\Theta)}(r, \theta) = \frac{1}{2\pi}e^{-r}$. Let $X = R\cos\Theta$ and $Y = R\sin\Theta$.

   (a) What is the joint density function of (X, Y)? [15 mins]

   (b) Are X and Y independent? [5 mins]

3. X is normally distributed with mean 0 and variance 1. Y is independent of X and has probability density functions

$$f_Y(y) = \frac{\lambda}{2}e^{-\lambda|y|}$$

   for some constant $\lambda > 0$. Find the joint density function of $Z = X + Y$ and $W = X + 3Y$. [10 mins]

4. Let X have a Poisson distribution with parameter λ, and given that $X = n$, let Y have a binomial distribution with parameters n and p.

   (a) What is the joint probability mass function of X and Y? [5 mins]

   (b) Show that the marginal distribution of Y is a Poisson distribution with parameter λp. [10 mins]

   (c) What is the conditional distribution of X conditional on $Y = i$? [10 mins]

5. Let X and Y be independent uniform random variables on intervals $[2, 4]$ and $[3, 4]$ respectively. Calculate the probability density function of $X+Y$. [10 mins]

6. The point (X, Y) is uniformly distributed on the set

$$\{(x, y)|x + y < 3, y > -1, x > -2\}$$

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(a) What is the conditional expectation of X given that $Y = 2$? [5 mins]

(b) What is $\mathrm{Cov}(X, Y)$? [10 mins]

7. If X is exponentially distributed with parameter λ_1 and Y is independent of X and normally distributed with mean μ and variance σ^2.

   (a) find the moment generating function of $X - Y$. [10 mins]

   (b) Use the Chernoff bound with $t = \frac{-2\mu}{\sigma^2}$ to obtain a *lower* bound on the probability that $X > Y$. [10 mins]

8. Let X have an exponential distribution with parameter λ. Suppose that given $X = x$, Y is normally distributed with mean x and variance σ^2.

   (a) What is the joint density function of X and Y? [5 mins]

   (b) What is the covariance of X and Y? [10 mins]

   (c) Conditional on $Y = 3$, what is the density function of X? [You do not need to calculate the constant factor.] [5 mins]

9. Consider the following experiment: Toss 14 fair coins. For each toss that results in a head, roll a fair (6-sided) die. Take the sum of the numbers rolled on all these dice. What is the expected outcome? [5 mins]

10. In a class of 100 students, the professor wants to determine what proportion of the students can answer a simple calculus question. The professor decides to test a random sample of 10 students. In fact 35 of the students could answer the question.

    (a) What is the expected number of students sampled who answer the question correctly? [5 mins]

    (b) What is the variance of the number of correct answers if the students are sampled with replacement, i.e. one student could be tested more than once? [10 mins]

    (c) What is the variance of the number of correct answers if the 10 students sampled must be 10 different students? [10 mins]

11. An ecologist is collecting snails. She collects a total of 40 snails. There are a total of 18 species of snail, and each snail is equally likely to be any of the species.

    (a) What is the expected number of species of snails she collects? [10 mins]

    (b) What is the variance of the number of species of snails she collects? [10 mins]

    (c) Using the one-sided Chebyshev inequality, find a bound on the probability that she collects all 18 species. [Hint: The Chebyshev inequality will give the probability of collecting at least 18 species.] [10 mins]

12. A car insurance company finds that of its claims, 70% are for accidents, and 30% are for thefts. The theft claims are all for \$20,000, while for the accident claims claim, the expected amount claimed is \$15,000, and the standard deviation of the amount claimed is \$30,000.

    (a) What are the expected amount claimed, and the standard deviation of the amount claimed for any claim? [10 mins]

    (b) If the company has \$16,700,000 available to cover claims, and receives 1000 claims, what is the probability that it is unable to cover the claims made? [10 mins]

13. You are considering an investment. You would originally invest \$1,000, and every year, the investment will either increase by 50% with probability 0.6 or decrease by 30% with probability 0.4. You plan to use the investment after 15 years. What is the expected value of the investment after 15 years? [10 mins]

14. A company is planning to run an advertising campaign. It estimates that the number of customers it gains from the advertising campain will be approximately normally distributed with mean 3,000 and standard deviation 300. It also estimates that the amount spent by each customer has expected value \$50 and standard deviation \$30.

    (a) Assuming the number of customers is large enough, what is the approximate distribution of the average amount spent per customer, conditional on the number of new customers being n. [5 mins]

    (b) What is the joint density function for the number of new customers and the average amount spent per customer? [10 mins]

15. A gambler is playing a slot machine in a casino. The gambler continues to bet \$1 each time. The machine has the following payouts:

Payout	Probability
\$1	0.4
\$10	0.03
\$100	0.002
\$1000	0.00005

    How many times does the gambler have to play before the probability of his having more money than he started with is less than 1%? [10 mins]