ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science) WINTER 2015

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Sample Final Examination
Model Solutions

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *An insurance company models claim sizes as having the following survival function*

$$S(x) = \frac{25(x+1)}{(x^2+2x+5)^2} \qquad x \geqslant 0$$

Calculate the TVaR of a loss at the 95% level.

The VaR is given by solving

$$\begin{aligned} S(x) &= 0.05 \\ \frac{25(x+1)}{(x^2+2x+5)^2} &= 0.05 \\ 25(x+1) &= 0.05(x^2+2x+5)^2 \end{aligned}$$

The solution is $x = 6.58959$. Now we calculate the $TVaR$ as

$$6.58959 + \frac{\int_{6.58959}^{\infty} \frac{25(x+1)}{(x^2+2x+5)^2} dx}{0.05}$$

We have

$$\begin{aligned} \int_{6.58959}^{\infty} \frac{x+1}{(x^2+2x+5)^2} dx &= \left[-\frac{1}{2(x^2+2x+5)} \right]_{6.58959}^{\infty} \\ &= \frac{1}{2(6.58959^2+2\times 6.58959+5)} \\ &= \frac{1}{123.2038} \\ &= 0.008116635 \end{aligned}$$

so the TVaR is $6.58959 + 500 \times 0.008116635 = 10.6479$.

2. *An insurance company models claim sizes as following a mixture of two distributions. With probability 0.3, claims follow a Weibull distribution with $\tau = 2$ and $\theta = 100$. With probability 0.7, claims follow a Pareto distribution with $\alpha = 3$ and $\theta = 250$.*

   (a) Which of the following is the VaR of the distribution at the 90% level?

   (i) 122.02

   (ii) 146.35

   (iii) 197.14

   (iv) 230.60

   The survival function is

$$S(x) = 0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3}$$

.

The VaR is the solution to

$$0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3} = 0.1$$

We try the values given:

x	$0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3}$
(i) 122.02	0.2801187
(ii) 146.35	0.2108953
(iii) 197.14	0.128501
(iv) 230.60	0.1000009

so (iv) 230.60 is the VaR.

(b) Calculate the TVaR at the 90% level.

The TVaR is given as

$$230.60 + \frac{\int_{230.60}^{\infty} 0.3e^{-\left(\frac{x}{100}\right)^2} + 0.7\frac{250^3}{(250+x)^3}\,dx}{0.1}$$

We have

$$\int_{230.60}^{\infty} e^{-\frac{x^2}{10000}}\,dx = 100\sqrt{\pi}\left(1-\Phi\left(\frac{230.60\sqrt{2}}{100}\right)\right) = 100\sqrt{\pi}(1-\Phi(3.261176)) = 0.0983279$$

and

$$\int_{230.60}^{\infty} \frac{250^3}{(250+x)^3}\,dx = \left[-\frac{250^3}{2(250+x)^2}\right]_{230.60}^{\infty} = \frac{250^3}{2(250+230.60)^2} = 33.82381$$

so the TVaR is

$$230.60 + \frac{0.3 \times 0.0983279 + 0.7 \times 33.82381}{0.1} = 467.66$$

3. *An insurance company observes the following sample of claims:*

   1.2,1.6,2.1,3.5, 3.7

   They use a Kernel density model with a uniform kernel with bandwidth 1. What is the probability that a claim is between 2.3 and 3.3?

   The probability that a claim is between 2.3 and 3.3 is

$$\frac{1}{5}\left((1-1)+(1-0.85)+(1-0.6)+(0.4-0)+(0.3-0)\right)=0.25$$

4. *An insurance company observes the following sample of claims:*

   2.3, 2.6, 2.8, 3.0, 3.5

   They use a Kernel density model with kernel a gamma distribution with $\alpha = 2$ *and* $\theta = \frac{x}{2}$ *(so the mean matches the observed data point). What is the variance of a random claim under this model?*

   The kernel density model is a mixture model, so the variance is given by the law of total variance:

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|I)) + \mathrm{Var}(\mathbb{E}(X|I))$$

   where I indicates which of the data points we are using. For a data point x, the expected value of the corresponding kernel is x, and the variance is $\frac{x^2}{2}$. We therefore have

$$\begin{aligned}\mathrm{Var}(X) =& \frac{1}{5}\left(\frac{2.3^2}{2}+\frac{2.6^2}{2}+\frac{2.8^2}{2}+\frac{3.0^2}{2}+\frac{3.5^2}{2}\right)+\frac{1}{5}\left(2.3^2+2.6^2+2.8^2+3.0^2+3.5^2\right)\\ &-\left(\frac{2.3+2.6+2.8+3.0+3.5}{5}\right)^2 = 4.2764\end{aligned}$$

5. *An insurance company observes the following sample of claims:*

   1.6, 2.2, 2.4, 3.1, 3.5

   They use a Kernel density model with a uniform kernel with bandwidth b. They calculate the VaR at the 90% level is 3.7. What bandwidth are they using?

   If the bandwidth is b, then the probability that a exceeds 3.7 is

$$\frac{1}{2}-\frac{1}{5}\left(\max\left(\frac{3.7-1.6}{2b},\frac{1}{2}\right)+\max\left(\frac{3.7-2.2}{2b},\frac{1}{2}\right)+\max\left(\frac{3.7-2.4}{2b},\frac{1}{2}\right)+\max\left(\frac{3.7-3.1}{2b},\frac{1}{2}\right)+\max\left(\right.\right.$$

   (Since 3.7 is larger than all the above values, there is no need to consider the possibility that the above values could be less than 0. In general, we would take the minimum of each value above with zero.)

We need to solve this equal to 0.1. For b small enough that only 3.5 contributes to the density at 3.7, this becomes

$$\frac{1}{2}-\frac{1}{5}\left(2+\frac{0.2}{2b}\right)=0.1$$

Multiplying by 10 and rearranging gives

$$\frac{0.4}{b}=0$$

which has no solutions in the interval. If we make $0.6 < b < 1.3$, so that only 3.5 and 3.1 contribute, the equation becomes

$$\frac{1}{2}-\frac{1}{5}\left(1.5+\frac{0.8}{2b}\right)=0.1$$

Multiplying by 10 and rearranging gives

$$\frac{0.8}{b}=1$$

which gives $b = 0.8$. This is in the interval, so this is the required bandwidth.

6. *An insurance company observes the following sample of claims:*

   1.4, 1.9, 2.8, 2.9, 3.6

   They use a Kernel density model with a Gaussian (normal) kernel with standard deviation 1. *What is the mode claim size (highest probability density)?*

   (i) 2.50238

   (ii) 2.69101

   (iii) 2.82243

   (iv) 2.94337

   The density function is

$$\frac{1}{5\sqrt{2\pi}}\left(e^{-\frac{(x-1.4)^2}{2}}+e^{-\frac{(x-1.9)^2}{2}}+e^{-\frac{(x-2.8)^2}{2}}+e^{-\frac{(x-2.9)^2}{2}}+e^{-\frac{(x-3.6)^2}{2}}\right)$$

   We want to find where this is maximised. The derivative is proportional to

$$(x-1.4)e^{-\frac{(x-1.4)^2}{2}}+(x-1.9)e^{-\frac{(x-1.9)^2}{2}}+(x-2.8)e^{-\frac{(x-2.8)^2}{2}}+(x-2.9)e^{-\frac{(x-2.9)^2}{2}}+(x-3.6)e^{-\frac{(x-3.6)^2}{2}}$$

   We try the values given:

x	$(x-1.4)e^{-\frac{(x-1.4)^2}{2}}+(x-1.9)e^{-\frac{(x-1.9)^2}{2}}+(x-2.8)e^{-\frac{(x-2.8)^2}{2}}+(x-2.9)e^{-\frac{(x-2.9)^2}{2}}+(x-$
(i) 2.50238	0.000051
(ii) 2.69101	2.35498
(iii) 2.82243	4.216045
(iv) 2.94337	6.250057

The mode is (i) $x = 2.50238$.

7. *An insurance company assigns a risk factor Θ to each individual. These Θ follow a gamma distribution with $\alpha = 2$ and $\theta = 400$. For an individual with risk factor $\Theta = \theta$, the size of a claim follows an inverse gamma distribution with $\alpha = 3$ and this value of θ. What is the probability that a random individual makes a claim in excess of \$3,000?*

   The probability that a claim from an individual with risk factor θ exceeds \$3,000 is

$$1 - e^{-\frac{\theta}{3000}}\left(1 + \frac{\theta}{3000} + \frac{\theta^2}{2\times 3000^2}\right)$$

   Therefore the probability that a claim from a randomly chosen individual exceeds \$3,000 is the expected value of this over the distribution of Θ. That is,

$$\begin{aligned}
&1 - \int_0^\infty e^{-\frac{\theta}{3000}}\left(1 + \frac{\theta}{3000} + \frac{\theta^2}{2\times 3000^2}\right)\frac{\theta}{400^2\Gamma(2)}e^{-\frac{\theta}{400}}\,d\theta\\
&= 1 - \int_0^\infty \frac{e^{-\frac{8.5\theta}{3000}}\left(\theta + \frac{\theta^2}{3000} + \frac{\theta^3}{2\times 3000^2}\right)}{400^2\Gamma(2)}\,d\theta\\
&= 1 - \frac{\left(\frac{3000}{8.5}\right)^2}{400^2} - \frac{2\left(\frac{3000}{8.5}\right)^3}{3000\times 400^2} - \frac{6\left(\frac{3000}{8.5}\right)^4}{2\times 3000^2\times 400^2}\\
&= 0.005938626
\end{aligned}$$

8. *An insurance company models the sizes of claims as a mixture distribution. Conditional on $\Theta = \theta$, the claim size has survival function*

$$S(x) = \frac{\theta^4}{x^4} \qquad x \geqslant \theta$$

   The distribution of Θ is a gamma distribution with $\alpha = 4$ and $\theta = 600$. For two claims which share a value of Θ, what is the probability that both are larger than \$7,000? [Hint: You will need to break into two cases: $\theta \leqslant 7000$ and $\theta > 7000$. You may use $\int_0^{7000} \frac{\theta^{11}e^{-\frac{\theta}{600}}}{600^{12}\Gamma(12)}\,d\theta = 0.4998014$.]

   For a fixed value of θ, the probability that two values are both larger than \$7,000 is $\left(\frac{\theta^4}{7000^4}\right)^2 = \frac{\theta^8}{7000^8}$, provided $\Theta < 7000$ and 1 if $\Theta > 7000$.

The probability of this is therefore the expectation of this over the distribution of Θ, which is

$$\int_0^{7000} \frac{\theta^8}{7000^8} \frac{\theta^3}{600^4\Gamma(4)} e^{-\frac{\theta}{600}}\,d\theta + P(\Theta > 7000)$$

The probability that $\Theta > 7000$ is

$$e^{-\frac{7000}{600}}\left(1 + \frac{7000}{600} + \frac{7000^2}{2\times 600^2} + \frac{7000^3}{6\times 600^3}\right) = 0.002961636$$

The first integral is

$$\frac{600^8\Gamma(12)}{7000^8\Gamma(4)}\int_0^{7000} \frac{\theta^{11}e^{-\frac{\theta}{600}}}{600^{12}\Gamma(12)}\,d\theta$$

where the quantity being integrated is the density of a gamma distribution with $\alpha = 12$ and $\theta = 600$. This probability is

$$1 - e^{-\frac{7000}{600}}\left(1 + \frac{7000}{600} + \cdots + \frac{7000^{11}}{600^{11}\times 10!}\right) = 0.4998014$$

The probability that $X > 7000$ is therefore $0.4998014\frac{600^8\Gamma(12)}{7000^8\Gamma(4)} + 0.002961636 = 0.01264949$.

9. *An insurance company models life expectancy as having hazard rate $\lambda(t) = \Theta e^{0.03t}$, where Θ varies between individuals following a normal distribution with $\mu = 0.0000013$ and $\sigma = 0.0000000021$. Calculate the probability that an individual aged 50 survives to age 80.*

   For a fixed value $\Theta = \theta$, the probability of surviving to age x is

$$S(x) = e^{-\theta\int_0^x e^{0.03t}\,dt} = e^{-\frac{e^{0.03x}-1}{0.03}\theta}$$

   The probability of a randomly chosen individual surviving to age x is the expected value of this over the distribution of Θ. That is $M_\Theta\left(-\frac{e^{0.03x}-1}{0.03}\right)$. Recall, for a normal distribution, we have $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, so

$$M_\Theta\left(-\frac{e^{0.03x}-1}{0.03}\right) = e^{-\frac{0.0000013(e^{0.03x}-1)}{0.03} + \frac{1}{2}0.0000000021^2\left(\frac{e^{0.03x}-1}{0.03}\right)^2}$$

   So the probability that an individual lives to age 50 is

$$S(50) = e^{-\frac{0.0000013(e^{1.5}-1)}{0.03} + \frac{1}{2}0.0000000021^2\left(\frac{e^{1.5}-1}{0.03}\right)^2} = 0.9998491$$

   and the probability that an individual lives to age 90 is

$$S(90) = e^{-\frac{0.0000013(e^{2.7}-1)}{0.03} + \frac{1}{2}0.0000000021^2\left(\frac{e^{2.7}-1}{0.03}\right)^2} = 0.9993987$$

so the probability that an individual currently aged 50 survives to age 90 is

$$\frac{0.9993987}{0.9998491} = 0.9995495$$

10. *An insurance company models life expectancy as having hazard rate* $\lambda(t) = \frac{\Theta}{120-t}$, *where* Θ *is uniformly distributed on the interval* $[0.05, 0.35]$. *Calculate the expected future lifetime of an individual aged 60.*

    The probability that an individual with $\Theta = \theta$ survives to age x is $e^{-\theta\left(\int_0^x \frac{1}{120-t}\,dt\right)} = e^{-\theta(\log(120)-\log(120-x))} = \left(\frac{120-x}{120}\right)^\theta$. The expected future lifetime for an individual aged 60 with $\Theta = \theta$ is therefore

$$\begin{aligned}\int_{60}^{120} \frac{\left(\frac{120-x}{120}\right)^\theta}{\left(\frac{120-60}{120}\right)^\theta}\,dx &= \int_{60}^{120} \left(\frac{120-x}{60}\right)^\theta dx\\ &= \int_0^{60} \frac{u^\theta}{60^\theta}\,du\\ &= \left[\frac{u^{\theta+1}}{60^\theta(\theta+1)}\right]_0^{60}\\ &= \frac{60}{\theta+1}\end{aligned}$$

    The expected future lifetime is therefore

$$\begin{aligned}\int_{0.05}^{0.35} \frac{60}{0.3(\theta+1)}\,d\theta = \int_{1.05}^{1.35} \frac{60}{0.3u}\,du = \left[\frac{60}{0.3}\log(u)\right]_{1.05}^{1.35} = \frac{60}{0.3}\log\left(\frac{1.35}{1.05}\right)\\ = 50.26289\end{aligned}$$

11. *An insurance company models an individual's lifetime as having density function* $f(x) = \begin{cases} 0.0000521566(12-x)^2 & \text{if } 0 < x < 10\\ 0.36e^{0.06x - e^{0.06x}} & \text{if } x > 10\end{cases}$ *Calculate the probability that an individual dies between ages 6 and 66.*

    The probability that an individual dies between ages 6 and 66 is

$$\begin{aligned}&0.0000521566\int_6^{10}(12-x)^2\,dx + \int_{10}^{66} 0.36e^{0.06x}e^{-e^{0.06x}}\,dx\\ &= \frac{0.0000521566}{3}\left[-(12-x)^3\right]_6^{10} + \left[-6e^{-e^{0.06x}}\right]_{10}^{66}\\ &= \frac{0.0000521566}{3}(216-8) + 6(e^{-e^{0.6}} - e^{-e^{3.96}}) = 0.9737131\end{aligned}$$

12. *An insurance company models the loss on a given policy as following a gamma distribution with* $\alpha = 3$ *and* $\theta = 1500$ *for values less than 15000,*

and following a Pareto distribution with $\alpha = 3$ and $\theta = 1200$ for values greater than 15000. 8% of claims are more than 15000. Calculate the variance of a random claim.

Conditional on being less than 15000, the expected value of a claim is

$$1500\frac{\int_0^{10} x^3e^{-x}\,dx}{\int_0^{10} x^2e^{-x}\,dx} = 4500\frac{\left(1-e^{-10}\left(1+10+\frac{10^2}{2}+\frac{10^3}{6}\right)\right)}{\left(1-e^{-10}\left(1+10+\frac{10^2}{2}\right)\right)} = 4465.854$$

and the expected value of the square is

$$1500^2\frac{\int_0^{10} x^4e^{-x}\,dx}{\int_0^{10} x^2\,dx} = 27000000\frac{\left(1-e^{-10}\left(1+10+\frac{10^2}{2}+\frac{10^3}{6}+\frac{10^4}{24}\right)\right)}{\left(1-e^{-10}\left(1+10+\frac{10^2}{2}\right)\right)} = 26281966$$

so the variance is

$$26281966 - 4465.854^2 = 6338114$$

Conditional on being more than 15000, the expected value is

$$\begin{aligned}15000+\left(\frac{16200}{1200}\right)^3\int_{15000}^{\infty}\left(\frac{1200}{1200+x}\right)^3 dx &= 15000+13.5^3\left[-\frac{1200^3}{2(1200+x)^2}\right]_{15000}^{\infty}\\ &= 15000+13.5^3\times\frac{1200^3}{2\times 16200^2}\\ &= 15000+8100 = 23100\end{aligned}$$

and the left-shifted second raw moment is

$$\begin{aligned}\left(\frac{16200}{1200}\right)^3\int_0^{\infty}\left(\frac{1200}{16200+x}\right)^3 2x\,dx &= 13.5^3\int_0^{\infty}1200^3\left(\frac{2}{(16200+x)^2}-\frac{32400}{(16200+x)^3}\right)dx\\ &= 13.5^3\times 1200^3\left[\frac{16200}{(16200+x)^2}-\frac{2}{(16200+x)}\right]_0^{\infty}\\ &= 16200^3\left(\frac{2}{16200}-\frac{16200}{16200^2}\right) = 262440000\end{aligned}$$

So the variance is

$$262440000 - 8100^2 = 196830000$$

The expected conditional variance is $0.08\times 196830000 + 0.92\times 6338114 =$ 21577465 and the variance of the conditional expectation is $0.08{\times}0.92\,(23100-4465.844)^2 =$ 25556258, so the variance is $21577465 + 25556258 = 47133723$.

13. *The number of incidents in a year follows a Poisson distribution with $\lambda = 5$. The number of claims resulting from an incident follows a negative binomial distribution with $r = 0.1$ and $\beta = 3.4$. Calculate the probability that there are exactly 3 claims in a given year.*

The distribution is a compound Poisson-Negative binomial. The pgf is $P(z) = e^{-5(1-(1+3.4-3.4z)^{-0.1})}$, so the probability of zero is $P(0) = e^{-5(1-4.4^{-0.1})} = 0.5023108$. We now use the recurrence relation

$$f_S(k) = \sum_{i=1}^{k} \frac{5i}{k} f_X(i) f_S(k-i)$$

We calculate

$$f_S(1) = 5 \times 0.1(4.4)^{-0.1}\left(\frac{3.4}{4.4}\right) \times 0.5023108 = 0.1673491$$

$$f_S(2) = \frac{5}{2} \times 0.1(4.4)^{-0.1}\left(\frac{3.4}{4.4}\right) \times 0.1673491 + 5 \times \frac{0.1 \times 1.1}{2}(4.4)^{-0.1}\left(\frac{3.4}{4.4}\right)^2 \times 0.5023108 = 0.09900028$$

$$f_S(3) = \frac{5}{3} \times 0.1(4.4)^{-0.1}\left(\frac{3.4}{4.4}\right) \times 0.09900028 + \frac{10}{3} \times \frac{1.1 \times 0.1}{2}(4.4)^{-0.1}\left(\frac{3.4}{4.4}\right)^2 \times 0.1673491 + 5 \times \frac{2.1 \times}{}$$

14. *The number of policies sold in a year follows a binomial distribution with $n = 100000$ and $p = 0.002$. The number of claims resulting from each policy sold follows a Poisson distribution with $\lambda = 0.02$. Calculate the variance of the total number of claims in a year.*

We can use the law of total variance $\mathrm{Var}(S) = \mathbb{E}(\mathrm{Var}(S|X)) + \mathrm{Var}(\mathbb{E}(S|X))$ where X is the number of policies sold. We have $\mathrm{Var}(S|X) = 0.02X$, and $\mathbb{E}(S|X) = 0.02X$, so

$$\mathrm{Var}(S) = 0.02\mathbb{E}(X) + 0.0004\,\mathrm{Var}(X) = 0.02 \times 100000 \times 0.002 + 0.0004 \times 100000 \times 0.002 \times 0.998 = 4.0784$$

15. *The number of fires follows a Poisson distribution with $\lambda = 5$, and the number of earthquakes follows a Poisson distribution with $\lambda = 0.4$. The number of claims resulting from a fire follows a negative binomial with $r = 1$ and $\beta = 1.2$. The number of claims resulting from an earthquake follows a Poisson distribution with $\lambda = 8$. Calculate the probability that there are exactly 2 claims in a given year.*

The sum of compound Poisson distributions is a compound Poisson distribution. The primary distribution is a Poisson distribution with $\lambda = 5.4$, and secondary distribution a mixture — with probability $\frac{5}{5.4}$ it is a negative binomial with $r = 1$ and $\beta = 1.2$, and with probability $\frac{0.4}{5.4}$ it is a Poisson with parameter 8. This mixture distribution has the following probabilities:

i	$f_X(i)$
0	0.4209003
1	0.2297672
2	0.1260143

We use the recurrence

$$f_S(k) = \sum_{i=1}^{k} \frac{5.4i}{k} f_X(i) f_S(k-i)$$

We calculate

$$f_S(0) = e^{-5.4(1-0.4209003)} = 0.04384308$$

$$\begin{aligned} f_S(1) &= 5.4 \times 0.2297672 \times 0.04384308 &= 0.05439799 \\ f_S(2) &= \frac{5.4}{2} \times 0.2297672 \times 0.05439799 + 5.4 \times 0.1260143 \times 0.04384308 &= 0.06358118 \end{aligned}$$

16. *An insurance company models the number of claims resulting from 1200 policies as following a compound Poisson-Poisson distribution with parameters 3 and 8. The following year, the company sells 1400 policies. What is the probability that there are exactly 2 claims the following year?*

The following year, the number of claims should be modelled as a compound Poisson-Poisson with parameters 3.5 and 8. This has p.g.f.

$$P(z) = e^{3.5(e^{8(z-1)}-1)}$$

We calculate

$$P(0) = e^{3.5(e^{-8}-1)} = 0.03023286$$

The recurrence relation is

$$f_S(k) = \sum_{i=1}^{k} \frac{3.5i}{k} e^{-8} \frac{8^i}{i!} f_S(k)$$

which gives

$$\begin{aligned} f_S(1) &= 3.5 \times 8e^{-8} \times 0.03023286 = 0.0002839758 \\ f_S(2) &= 1.75 \times 8e^{-8} \times 0.0002839758 + 3.5 \times 32e^{-8} \times 0.03023286 = 0.001137237 \end{aligned}$$

17. *An insurance company models the number of claims resulting from 1500 policies as following a compound Poisson-Poisson distribution with parameters 5 and 2. How many policies should the company sell the following year in order to make the probability of receiving at least 2 claims at least 0.99?*

    (i) 1423

    (ii) 1641

    (iii) 1950

    (iv) 2274

    If the company sells N policies the following year, the distribution of the number of claims is a compound Poisson-Poisson distribution with parameters $\frac{N}{300}$ and 2. The probability of zero claims is $e^{\frac{N}{300}(e^{-2}-1)}$, and the probability of one claim is $\frac{N}{300} \times 2e^{-2} \times e^{\frac{N}{300}(e^{-2}-1)}$. We want to ensure that the probability of zero or one claim is at most 0.01. That is:

$$e^{\frac{N}{300}(e^{-2}-1)}\left(1+2e^{-2}\frac{N}{300}\right) \leqslant 0.01$$
$$e^{-0.002882216N}(1+0.0009022352N) \leqslant 0.01$$

    Evaluating the options given, we get:

N	$e^{-0.002882216N}(1+0.0009022352N)$
(i) 1423	0.037797311
(ii) 1641	0.021900870
(iii) 1950	0.009998473
(iv) 2274	0.004346145

    so (iii) $N = 1950$ is the number needed.

18. *An insurance company models loss size as following a Pareto distribution with $\alpha = 4$ and $\theta = 6000$. The company introduces a deductible of \$1,000. Calculate the expected payment per claim after the deductible is introduced.*

    The probability that a loss exceeds the deductible is $S(1000) = \left(\frac{6000}{6000+1000}\right)^4 = 0.5397751$. The survival function conditional on exceeding the deductible is

$$\frac{S(x)}{S(1000)} = \frac{\left(\frac{6000}{6000+x}\right)^4}{\left(\frac{6000}{7000}\right)^4} = \left(\frac{7000}{6000+x}\right)^4$$

    so the loss per claim after the deductible is applied follows a Pareto distribution with $\alpha = 4$ and $\theta = 7000$. The expected payment per claim is therefore $\frac{7000}{4-1} = \$2,333.33$.

19. *An insurance company models loss size as following a Weibull distribution with $\tau = 2$ and $\theta = 2000$. The company wants to introduce a deductible so that the expected payment per loss is \$1400. What deductible should it introduce?*

With a deductible of d, the expected payment per loss is given by

$$\begin{aligned}\int_d^\infty S(x)\,dx &= \int_d^\infty e^{-\left(\frac{x}{2000}\right)^2}\,dx = 2000\sqrt{\pi}\int_d^\infty \frac{1}{\left(\sqrt{2\pi}\frac{2000}{\sqrt{2}}\right)}e^{-\frac{x^2}{2000^2}}\,dx\\ &= 2000\sqrt{\pi}\left(1-\Phi\left(\frac{d}{1000\sqrt{2}}\right)\right)\end{aligned}$$

Setting this equal to \$1400 gives

$$\begin{aligned}2000\sqrt{\pi}\left(1-\Phi\left(\frac{d}{1000\sqrt{2}}\right)\right) &= 1400\\ \Phi\left(\frac{d}{1000\sqrt{2}}\right) &= 1-\frac{1400}{2000\sqrt{\pi}} = 0.6050673\\ \frac{d}{1000\sqrt{2}} &= 0.2664854\\ d &= 376.87\end{aligned}$$

20. *An insurance company models loss size as following a log-logistic distribution distribution with $\gamma = 2$ and $\theta = 2000$. The company wants to introduce a deductible with loss elimination ratio 30%.*

(a) What deductible should it introduce?

The mean of the log-logistic distribution is $2000\Gamma(1.5)\Gamma(0.5) = 1000\pi$, so to obtain a loss elimination ratio of 30%, the expected payment per loss after the deductible must be 700π.

The expected payment per loss after a deductible of d is introduced is

$$\begin{aligned}\int_d^\infty S(x)\,dx &= \int_d^\infty \frac{2000^2}{2000^2+x^2}\,dx = 2000\int_{\frac{d}{2000}}^\infty \frac{1}{1+x^2}\,dx\\ &= 2000\left[\tan^{-1}x\right]_{\frac{d}{2000}}^\infty = 2000\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{d}{2000}\right)\right)\end{aligned}$$

We therefore want to solve

$$
\begin{aligned}
2000\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{d}{2000}\right)\right) &= 700\pi\\
2000\tan^{-1}\left(\frac{d}{2000}\right) &= 1000\pi-700\pi=300\pi\\
\tan^{-1}\left(\frac{d}{2000}\right) &= \frac{300\pi}{2000}=0.15\pi\\
\frac{d}{2000} &= \tan(0.15\pi)\\
d &= 2000\tan(0.15\pi)=1019.05
\end{aligned}
$$

(b) In the following years, there is uniform inflation of 4% every year. How many years does it take until the deductible calculated in (a) gives a loss elimination ratio of less than 25%?

If we discount for inflation, after n years, the deductible is equivalent to $1019.05(1.04)^{-n}$. We want to determine when this gives a loss elimination ration of 25%. As in part (a), a loss elimination ratio of 25% means the expected payment per loss is $\$750\pi$, so to get the deductible, we solve:

$$
\begin{aligned}
2000\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{d}{2000}\right)\right) &= 750\pi\\
2000\tan^{-1}\left(\frac{d}{2000}\right) &= 1000\pi-750\pi=250\pi\\
\tan^{-1}\left(\frac{d}{2000}\right) &= \frac{250\pi}{2000}=0.125\pi\\
\frac{d}{2000} &= \tan(0.125\pi)\\
d &= 2000\tan(0.125\pi)=828.43
\end{aligned}
$$

We therefore want to solve

$$
\begin{aligned}
828.43 &= 1019.05(1.04)^{-n}\\
1.04^n &= \frac{1019.05}{828.43}\\
n &= \frac{\log\left(\frac{1019.05}{828.43}\right)}{\log(1.04)}=5.280325
\end{aligned}
$$

So the loss elimination ratio is below 25% after 6 years.

21. *Losses follow a generalised Pareto distribution with $\alpha=2$, $\tau=3$, and $\theta=3000$. An insurance company introduces a deductible of \$600. Calculate the loss elimination ratio of this deductible after inflation of 12%.*

After inflation of 12%, losses follow a generalised Pareto distribution with $\alpha = 2$, $\tau = 3$ and $\theta = 3360$. The expected loss is therefore $3360\frac{3}{2-1} = 10080$. After the deductible, the expected payment per loss is

$$\begin{aligned}
\int_{600}^{\infty}(x-600)\frac{\Gamma(5)}{\Gamma(2)\Gamma(3)}\frac{\left(\frac{x}{3360}\right)^3}{x\left(1+\frac{x}{3360}\right)^5}\,dx &= 12\times 3360^2\int_{600}^{\infty}\frac{x^3-600x^2}{(3360+x)^5}\,dx\\
&= 12\times 3360^2\int_{3960}^{\infty}((u-3360)^3-600(u-3360)^2)u^{-5}\,du\\
&= 12\times 3360^2\int_{3960}^{\infty}u^{-2}-10680u^{-3}+11280\times 3360u^{-4}-3960\times\\
&= 12\times 3360^2\left[-u^{-1}+5340u^{-2}-3760\times 3360u^{-3}+990\times 3360^2\right.\\
&= 9482.087
\end{aligned}$$

The loss elimination ratio is therefore $1-\frac{9482.087}{10080} = \frac{597.91}{10080} = 0.05931677$

22. *Losses follow an inverse Pareto distribution with $\tau = 4$ and $\theta = 6000$.*

(a) Calculate the expected payment per claim with a policy limit of \$1,000,000.

The expected payment per claim is given by

$$\begin{aligned}
&\int_0^{1000000} 1-\left(\frac{x}{x+6000}\right)^4 dx\\
&= 1000000-\int_0^{1000000}\left(\frac{x}{x+6000}\right)^4 dx\\
&= 1000000-\int_{6000}^{1006000}\left(\frac{u-6000}{u}\right)^4 dx\\
&= 1000000-\int_{6000}^{1006000} 1-24000u^{-1}+6\times 6000^2u^{-2}-4\times 6000^3u^{-3}+6000^4u^{-4}\,du\\
&= \left[24000\log(u)+6\times 6000^2u^{-1}-2\times 6000^3u^{-2}+\frac{6000^4}{3}u^{-3}\right]_{6000}^{1006000}\\
&= 24000\log\left(\frac{1006000}{6000}\right)+\frac{6\times 6000^2}{1006000}-\frac{6\times 6000^2}{6000}-\frac{2\times 6000^3}{1006000^2}+\frac{2\times 6000^3}{6000^2}+\frac{6000^4}{3\times 1006000^3}-\frac{6000^4}{3\times 6000^3}\\
&= \$97,141.75
\end{aligned}$$

(b) Calculate the expected payment per claim if there is 15% inflation (the policy limit remains at \$1,000,000.)

If there is 15% inflation, the loss distribution is an inverse Pareto distribution with $\tau = 4$ and $\theta = 6900$. The expected payment per claim is therefore

$$\begin{aligned}
&\int_0^{1000000} 1 - \left(\frac{x}{x+6900}\right)^4 dx \\
&= 1000000 - \int_0^{1000000} \left(\frac{x}{x+6900}\right)^4 dx \\
&= 1000000 - \int_{6900}^{1006900} \left(\frac{u-6900}{u}\right)^4 dx \\
&= 1000000 - \int_{6900}^{1006900} 1 - 27600u^{-1} + 6\times 6900^2u^{-2} - 4\times 6900^3u^{-3} + 6900^4u^{-4}\,du \\
&= \left[27600\log(u) + 6\times 6900^2u^{-1} - 2\times 6900^3u^{-2} + \frac{6900^4}{3}u^{-3}\right]_{6900}^{1006900} \\
&= 27600\log\left(\frac{1006900}{6900}\right) + \frac{6\times 6900^2}{1006900} - \frac{6\times 6900^2}{6900} - \frac{2\times 6900^3}{1006900^2} + \frac{2\times 6900^3}{6900^2} + \frac{6900^4}{3\times 1006900^3} - \frac{6900^4}{3\times 6900^3} \\
&= \$107,916.90
\end{aligned}$$

23. *Losses follow an exponential distribution with $\theta = 7000$. There is a deductible of \$700, a policy limit of \$25,000 and coinsurance such that the insurance pays 80% of the claim after the policy limit and deductible have been applied. Calculate the expected payment per claim and the variance of the payment per claim.*

The expected payment per claim is

$$\begin{aligned}
0.8\int_{700}^{25000} \frac{e^{-\frac{x}{7000}}}{e^{-\frac{700}{7000}}}\,dx &= 0.8\int_0^{24300} e^{-\frac{u}{7000}}\,du \\
&= 5600\left(1 - e^{-\frac{24300}{7000}}\right) = \$5425.99
\end{aligned}$$

The expected value of the square of the payment per claim is given by

$$\begin{aligned}
0.64\int_0^{24300} 2xe^{-\frac{x}{7000}}\,dx &= 0.64\left([-14000xe^{-\frac{x}{7000}}]_0^{24300} + 14000\int_0^{24300} e^{-\frac{x}{7000}}\,dx\right) \\
&= 8960\left(7000(1-e^{-\frac{24300}{7000}}) - 24300e^{-\frac{24300}{7000}}\right) &= 54005749
\end{aligned}$$

so the variance is $54005749 - 5425.99^2 = 24564349$

24. *Losses follow a Pareto distribution with $\alpha = 3$ and $\theta = 5000$. There is a deductible of \$1000. The insurance company wants to reduce the TVaR (per claim) for this policy at the 99.9% level to \$60,000. What policy limit should they set?*

The VaR at the 99.9% level is the solution to

$$\begin{aligned}
\frac{\left(\frac{5000}{5000+(x+1000)}\right)^3}{\left(\frac{5000}{5000+1000}\right)^3} &= 0.001 \\
\frac{5000}{5000+(x+1000)} &= 0.1 \times \frac{5}{6} \\
\frac{x+1000}{5000} + 1 &= 12 \\
x &= 54000
\end{aligned}$$

The TVaR with a policy limit of u is therefore

$$\begin{aligned}
&54000 + \frac{\int_{54000}^{u-1000} \left(\frac{6000}{5000+x+1000}\right)^3 dx}{0.001} \\
&= 54000 + 1000 \int_{60000}^{u+5000} 6000^3 s^{-3}\, ds \\
&= 54000 + 500[-6000^3 s^{-2}]_{60000}^{u+5000}\, ds \\
&= 54000 + \frac{500 \times 6000^3}{60000^2} - \frac{500 \times 6000^3}{(u+5000)^2} \\
&= 71361.11 - \frac{500 \times 6000^3}{(u+5000)^2}
\end{aligned}$$

The policy limit is therefore found by solving

$$\begin{aligned}
71361.11 - \frac{500 \times 6000^3}{(u+5000)^2} &= 60000 \\
\frac{500 \times 6000^3}{(u+5000)^2} &= 11361.11 \\
(u+5000)^2 &= \frac{500 \times 6000^3}{11361.11} \\
u &= \sqrt{\frac{500 \times 6000^3}{11361.11}} - 5000 = \$92,499.30
\end{aligned}$$

25. *Losses follow a Weibull distribution with $\tau = 3$ and $\theta = 4000$. Loss frequency follows a Negative binomial distribution with $r = 6$ and $\beta = 4$.*

The insurance company wants to reduce the expected number of claims to 22. What deductible should it introduce in order to achieve this?

The expected number of losses is $r\beta = 24$, so the probability that a loss leads to a claim should be $\frac{22}{24}$. The deductible is therefore found by solving

$$\begin{aligned}
S(x) &= \frac{22}{24} \\
e^{-\left(\frac{d}{4000}\right)^3} &= \frac{22}{24} \\
\left(\frac{d}{4000}\right)^3 &= -\log\left(\frac{22}{24}\right) = \log\left(\frac{24}{22}\right) \\
\frac{d}{4000} &= \sqrt[3]{\log\left(\frac{24}{22}\right)} \\
d = 4000\sqrt[3]{\log\left(\frac{24}{22}\right)} &= \$1,772.50
\end{aligned}$$

26. *Losses follow a gamma distribution with $\alpha = 2$ and $\theta = 2000$. Loss frequency follows a Poisson distribution with $\lambda = 4$. If the company introduces a deductible of \$600, what is the probabilitiy that it receives more than 2 claims after the deductible?*

    The probability that a loss results in a claim after the deductible is

    $$\int_{600}^{\infty} \frac{xe^{-\frac{x}{2000}}}{2000^2}\,dx = e^{-\frac{600}{2000}}\left(1 + \frac{600}{2000}\right) = 1.3e^{-0.3} = 0.9630637$$

    The claim frequency is therefore a Poisson distribution with parameter $4 \times 1.3e^{-0.3}$, so the probability of receiving more than 2 claims is

    $$1 - e^{-5.2e^{-0.3}}\left(1 + 5.2e^{-0.3} + \frac{(5.2e^{-0.3})^2}{2}\right) = 0.7394392$$

27. *Losses follow a log-logistic distribution with $\gamma = 2$ and $\theta = 6000$. Claim frequency with a deductible of \$2000 is modelled as a negative binomial with $r = 5$ and $\beta = 2.6$. what would be the distribution of the claim frequency if the deductible were removed?*

    The probability that a loss leads to a claim is

    $$S(2000) = \frac{1}{\left(1 + \left(\frac{2000}{6000}\right)^2\right)} = 0.9$$

    If $P(z)$ is the p.g.f. of the number of losses, then we have

    $$P(0.1 + 0.9z) = (3.6 - 2.6z)^{-5}$$

substituting $w = 0.1 + 0.9z$, which gives $z = \frac{10w-1}{9}$, we get

$$P(w) = \left(3.6 - \frac{26w - 2.6}{9}\right)^{-5} = \left(\frac{35}{9} - \frac{26}{9}w\right)^{-5}$$

This is the p.g.f. of a negative binomial with $\beta = \frac{26}{9}$, and $r = 5$.