ACSC/STAT 3703, Actuarial Models I

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Toby Kenney

Homework Sheet 8

Model Solutions

1. *An insurance company has the following portfolio of fire insurance policies:*

Type of policy	*Number*	*Probability of claim*	*mean claim*	*standard deviation*
Retail	*350*	*0.0242*	*\$3,522*	*\$4,820*
Office	*631*	*0.0110*	*\$2,710*	*\$9,024*
Manufacture	*402*	*0.0524*	*\$8,015*	*\$14,254*

They model aggregate losses using an inverse gamma distribution. Calculate the cost of reinsuring losses above \$500,000, if there is a 30% loading on the reinsurance premium.

We calculate the mean and variance of the aggregate loss:

Type of policy	$\mathbb{E}(N)$	$\mathrm{Var}(N)$	mean aggregate loss	var aggregate loss
Retail	8.47	8.265026	29831.34	299301810.777
Office	6.941	6.864649	18810.11	615638178.737
Manufacture	21.0648	19.96100448	168834.372	5562172093.26
Total			217475.822	6477112082.77

Using an inverse gamma approximation, the method of moments gives the following parameters

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= 217475.822\\
\frac{\theta^2}{(\alpha-1)^2(\alpha-2)} &= 6477112082.77\\
\alpha-2 &= \frac{217475.822^2}{6477112082.77} = 7.30197849755\\
\alpha &= 9.30197849755\\
\theta &= 1805479.59798
\end{aligned}$$

The expected reinsurance payment is therefore

$$\begin{aligned}\int_d^\infty (x-d)\frac{\theta^\alpha e^{-\frac{\theta}{x}}}{x^{\alpha+1}\Gamma(\alpha)}\,dx &= \int_d^\infty \frac{\theta^\alpha e^{-\frac{\theta}{x}}}{x^{\alpha}\Gamma(\alpha)}\,dx - d\int_d^\infty \frac{\theta^\alpha e^{-\frac{\theta}{x}}}{x^{\alpha+1}\Gamma(\alpha)}\,dx \\ &= \frac{\theta}{\alpha-1}\int_d^\infty \frac{\theta^{\alpha-1} e^{-\frac{\theta}{x}}}{x^{\alpha}\Gamma(\alpha-1)}\,dx - d\int_d^\infty \frac{\theta^\alpha e^{-\frac{\theta}{x}}}{x^{\alpha+1}\Gamma(\alpha)}\,dx \\ &= 776.306\end{aligned}$$

With a 30% loading The premium is therefore $776.306 \times 1.3 = \$1009.20$.

2. *An insurance company sells medical malpractice insurance. It estimates that the standard deviation of the aggregate annual claim is \$420,000 and the mean is \$28,000.*

   (a) How many years history are needed for a clinic to be assigned full credibility? (Use $r = 0.05$, $p = 0.99$.)

   The coefficient of variation for aggregate annual claim is $\frac{420000}{28000} = 15$. For the average of n years of aggregate claims, the coefficient of variation is $\frac{15}{\sqrt{n}}$. Using $r = 0.05$ and $p = 0.99$, the standard for full credibility is obtained by solving:

$$\begin{aligned}P\left(\left|\frac{\overline{X}-\mu}{\mu}\right| < 0.05\right) &> 0.99 \\ 2\Phi\left(\frac{0.05\sqrt{n}}{15}\right) - 1 &> 0.99 \\ \frac{\sqrt{n}}{300} &> 2.575829 \\ n &> (300 \times 2.575829)^2 \\ &= 597140.553352\end{aligned}$$

   so 597141 years are needed.

   The standard net premium for this policy is \$28000. A clinic has claimed a total of \$92,032 in the last 24 years.

   (b) What is the net Credibility premium for this company, using limited fluctuation credibility?

   The credibility of 24 years of experience is $Z = \sqrt{\frac{24}{597140.553352}} = 0.00633968000925$. The premium for this company is therefore $0.00633968000925 \times \frac{92032}{24} + 0.993660319991 \times 28000 = \$27846.80$.

Standard Questions

3. *An auto insurer divides drivers into two categories: Safe and Dangerous. The number of claims made by a policyholder follows a Poisson distribution with a certain mean λ, depending on the type of driver. The characteristics of each type of policy are given in the following table.*

Category	λ	*mean claim*	*standard deviation of claim*
Safe	*0.01*	*14309*	*293054*
Dangerous	*0.03*	*25234*	*402346*

The insurer sells a total of 800 policies. The insurer buys stop-loss reinsurance from a reinsurer which models aggregate losses as following a Pareto distribution. The reinsurer charges a loading of 30%. If the attachment point is set equal to 1.5 times average aggregate losses, then the reinsurance premium is equal to 50.25% of expected aggregate losses. How many of each policy type does the insurer insure?

Using a Pareto distribution for aggregate losses, the expected aggregate losses are $\theta\alpha - 1$, and the expected payment on a reinsurance policy with attachment point a is

$$\begin{aligned}\int_a^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx &= \theta^\alpha \int_{a+\theta}^\infty u^{-\alpha}\,du \\ &= \theta^\alpha\left[-\frac{u^{1-\alpha}}{\alpha-1}\right]_{a+\theta}^\infty \\ &= \frac{\theta^\alpha}{(\alpha-1)(\theta+a)^{\alpha-1}}\end{aligned}$$

In particular, if $a = 1.5\frac{\theta}{\alpha-1}$, then this becomes

$$\frac{\theta^\alpha}{(\alpha-1)\left(\theta+1.5\frac{\theta}{\alpha-1}\right))^{\alpha-1}} = \frac{\theta}{(\alpha-1)\left(1+\frac{1.5}{\alpha-1}\right)^{\alpha-1}}$$

We are given that

$$1.3\frac{\theta}{(\alpha-1)\left(1+\frac{1.5}{\alpha-1}\right)^{\alpha-1}} = 0.5025\frac{\theta}{\alpha-1}$$

Which gives

$$\left(1+\frac{1.5}{\alpha-1}\right)^{\alpha-1} = 2.587065$$

Numerically, this is solved by $\alpha = 2.1151$, which gives $\frac{\alpha-2}{\alpha} = 0.05441823$

If the number of claims for each policyholder follows a Poisson distribution, then the number of aggregate claims for each class also follows a Poisson distribution. The mean and variance of the aggregate claims are both proportional to λ. Let m and n be the number of policies sold to safe and dangerous drivers

Category	λ	$\mathbb{E}(S)$	$\mathrm{Var}(S)$
Safe	$0.01m$	$143.09m$	$860853943.97m$
Dangerous	$0.03n$	$757.02n$	$4875571754.16n$
Total		$143.09m + 757.02n$	$860853943.97m + 4875571754.16n$

Setting the aggregate mean and variance as the mean and variance of a Pareto distribution gives

$$\begin{aligned}
\frac{\theta}{\alpha - 1} &= 143.09m + 757.02n \\
\frac{\alpha\theta}{(\alpha-1)^2(\alpha-2)} &= 860853943.97m + 4875571754.16n \\
\frac{\alpha - 2}{\alpha} &= \frac{(143.09m + 757.02n)^2}{860853943.97m + 4875571754.16n}
\end{aligned}$$

So m and n must satisfy

$$\frac{(143.09m + 757.02n)^2}{860853943.97m + 4875571754.16n} = 0.05441823$$

which gives

$$(143.09m+757.02n)^2 = 0.05441823(860853943.97m+4875571754.16n) = 46846147.9194m+265319985.099$$

Substituting $n = 800 - m$, we get

$$\begin{aligned}
(143.09m + 757.02(800 - m))^2 &= 46846147.9194m + 265319985.099(800 - m) \\
(605616 - 613.93m)^2 &= 212255988079 - 218473837.18m \\
366770739456 - 743611661.76m + 376910.0449m^2 &= 212255988079 - 218473837.18m \\
376910.0449m^2 - 525137824.58m + 154514751377 &= 0 \\
m &= \frac{525137824.58 \pm \sqrt{525137824.58^2 - 4 \times 376910.0449}}{2 \times 376910.0449} \\
&= 696.635485954 \pm 274.499054346 \\
m &= 422.136431608
\end{aligned}$$

So they sell policies to 422 safe drivers and 378 dangerous drivers.

4. *A home insurance company sets the standard for full credibility as 4142 policy-years. The book estimates are 0.09 claims per policy-year for claim frequency and $2,038 per claim for claim severity.*

The company changes the standard to 3922 policy-years for frequency and 408 claims for severity. For a particular policyholder with 28 policy-years of experience, who made 8 claims in that time, this results in a 3% increase in premiums. What was the total amount claimed by this policyholder?

Let x be the total amount claimed. Using the old standard, the credibility of 28 policy-years is $Z = \sqrt{\frac{28}{4142}} = 0.0822193366207$, so the policyholder's premium was

$$0.0822193366207\frac{x}{28}+0.917780663379\times 0.09\times 2038 = 0.00293640487931x+168.339329277$$

Since the change resulted in a 3% increase, the premium after the change is

$$1.03(0.00293640487931x+168.339329277) = 0.00302449702569x+173.389509155$$

Under the new standard, the credibility of 28 policy-years is $Z = \sqrt{\frac{28}{3922}} = 0.0844938736618$, so the credibility estimate for the number of claims per year by this policyholder is $0.0844938736618\frac{8}{28} + 0.915506126338 \times 0.09 = 0.106536658131$. The credibility of 8 claims is $Z = \sqrt{\frac{8}{408}} = 0.140028008403$, so the credibility estimate for average loss per claim for this policyholder is

$$0.140028008403\times\frac{x}{8}+0.859971991597\times 2038 = 0.0175035010504x+1752.62291887$$

Thus, we need to solve

$$\begin{aligned}0.00302449702569x + 173.389509155 &= 0.106536658131\,(0.0175035010504x + 1752.62291887)\\ &= 0.0018647645075x + 186.71858874\\ 0.00115973251819x &= 13.329079585\\ x &= \$11,493.24\end{aligned}$$