ACSC/STAT 3720, Life Contingencies I
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Homework Sheet 1
Model Solutions

Basic Questions

1. *An insurance company models the future lifetime of an individual as having survival function* $S(x) = e^{-\left(\frac{x}{80}\right)^2}$. *Calculate*

   (a) Force of mortality.

   Force of mortality is given by

$$\mu_x = -\frac{S'(x)}{S(x)} = \frac{2xe^{-\left(\frac{x}{80}\right)^2}}{80^2 e^{-\left(\frac{x}{80}\right)^2}} = \frac{2}{80^2}x = \frac{x}{3200}$$

   (b) The mean and standard deviation of T_x

   The mean of T_x is given by

$$\int_0^\infty S(x+t)dt = \int_0^\infty e^{-\left(\frac{x+t}{80}\right)^2} dt$$

   Now we have that $\frac{1}{80\sqrt{\pi}}e^{-\left(\frac{x+t}{80}\right)^2}$ is the density function of a normal distribution with mean $-t$ and variance 3200. Therefore

$$\int_0^\infty e^{-\left(\frac{x+t}{80}\right)^2} dt = 80\sqrt{\pi}\left(1 - \phi\left(\frac{x}{40\sqrt{2}}\right)\right)$$

   The expectation of ${T_x}^2$ is given by $\int_0^\infty 2te^{-\left(\frac{x+t}{80}\right)^2} dt = \int_0^\infty 2(x+t)e^{-\left(\frac{x+t}{80}\right)^2} dt - 2x\mathbb{E}(T_x)$. Now $\int 2(x+t)e^{-\left(\frac{x+t}{80}\right)^2} dt = -80^2 e^{-\left(\frac{x+t}{80}\right)^2} + C$, so $\mathbb{E}({T_x}^2) = 6400e^{-\left(\frac{x}{80}\right)^2} - 2\mathbb{E}(T_x)$. The standard deviation of T_x is therefore given by

$$\sqrt{6400e^{-\left(\frac{x}{80}\right)^2} - 160\sqrt{\pi}\left(1 - \phi\left(\frac{x}{40\sqrt{2}}\right)\right) - 6400\pi\left(1 - \phi\left(\frac{x}{40\sqrt{2}}\right)\right)^2}$$

   (c) The mean curtate future lifetime.

   The probability that the future lifetime is between n and $n+1$ years is

$${}_nP_x - {}_{n+1}p_x = e^{-\left(\frac{x+n}{80}\right)^2} - e^{-\left(\frac{x+n+1}{80}\right)^2}$$

   so the mean curtate future lifetime is

$$\sum_{n=0}^\infty n\left(e^{-\left(\frac{x+n}{80}\right)^2} - e^{-\left(\frac{x+n+1}{80}\right)^2}\right) = \sum_{n=1}^\infty e^{-\left(\frac{x+n}{80}\right)^2}$$

2. *An insurance company uses a survival model with survival function* ${}_tp_x = \left(1 - \frac{t}{\omega - x}\right)^{\frac{1}{8}}$*, where* ω *is the maximum age attainable. The company wants to ensure that the life expectancy of an individual aged 65 under this model is 16 years. What age should they choose as the maximum age attainable?*

   For a given value of ω, the life expectancy of an individual aged 65 is given by $\int_0^{\omega-65} {}_tp_{65}dt = \int_0^{\omega-65}\left(\frac{s}{\omega-65}\right)^{\frac{1}{8}}ds$ where $s = \omega - x - t$. This evaluates to $\frac{1}{1.125} \times \frac{(\omega-65)^{1.125}}{(\omega-65)^{0.125}} = \frac{8}{9}(\omega - 65)$. Therefore, they should choose $\omega = 65 + \frac{9}{8} \times 16 = 83$.

3. *An insurance company uses a survival model given by*

$$S_0(x) = \frac{2}{3}\left(1 - \frac{x}{100}\right)^{\frac{1}{6}} + \frac{1}{3}\left(1 - \frac{x}{125}\right)^{\frac{1}{4}}$$

   Using this model, prepare a life table for the ages from 35 to 40, using radix 10,000.

   We calculate

x	$S_0(x)$
35	0.9275317
36	0.9250740
37	0.9225880
38	0.9200730
39	0.9175282
40	0.9149528

   From this, we calculate $l_x = l_{35}\frac{S_0(x)}{S_0(35)}$, where we are given $l_{35} = 10000$, so we calculate

x	l_x	d_x
35	10000.00	26.50
36	9973.50	26.80
37	9946.70	27.11
38	9919.59	27.44
39	9892.15	27.77
40	9864.38	28.11

4. *Using the lifetable:*

x	l_x	d_x
65	*10000.00*	*30.89*
66	*9969.11*	*33.45*
67	*9935.66*	*36.23*
68	*9899.43*	*39.22*
69	*9860.21*	*42.46*
70	*9817.75*	*45.95*
71	*9771.80*	*49.71*
72	*9722.09*	*53.76*

calculate the probability that an individual aged 65 years and two months survives another 5 years, using:

(a) the uniform distribution of deaths assumption.

The probability that an individual aged 65 survives 2 months is $1 - \frac{1}{6} \times 0.003089 = 0.999485$. The probability that an individual aged 65 dies between ages 70 and 70 years and 2 months is $\frac{1}{6} \times 0.004595 = 0.000765.83$. Therefore the probability that an individual aged 65 survives to age 70 and 2 months is $0.981775 - 0.00076583 = 0.981009$. The probability that an individual aged 65 and 2 months survives for 5 years is therefore $\frac{0.981009}{0.999485} = 0.9815143$.

(b) the constant force of mortality assumption.

Using constant force of mortality, the mortality rate μ_{65} between ages 65 and 66 satisfies $e^{-\mu_{65}} = 0.996911$. At this mortality rate, the probability an individual survives 2 months is $e^{-\frac{\mu_{65}}{6}} = 0.996911^{\frac{1}{6}} = 0.999484503$. Similarly, the mortality rate between ages 70 and 71 satisfies $e^{-\mu_{65}} = \frac{9771.80}{9817.75}$, so the probability of an individual aged 70 surviving 2 months is $\left(\frac{9771.80}{9817.75}\right)^{\frac{1}{6}} = 0.999218425$. The probability of an individual aged 65 surviving to 70 and 2 months is therefore $0.981775 \times 0.999218425 = 0.9810076692$, and the probability of an individual aged 65 and 2 months surviving another 5 years is $\frac{0.9810076692}{0.999484503} = 0.9815136$.

Standard Questions

5. *An insurance company wants to use a model of mortality of the form $\mu_x = \frac{a}{b-x}$. Based on the company's data, the life expectancy for an individual aged 65 is 19 years, and the probability of an individual aged 44 surviving for 10 years is 0.982. The company chooses the values of a and b to ensure that these observations are matched by the model. Which of the following is the correct value of a? Justify your answer.*

   (i) 0.0443

   (ii) 0.0654

   (iii) 0.102

   (iv) 0.288

   We have $S_x(t) = e^{-\int_0^t \mu_{x+s}ds}$, and we have $\int_0^t \frac{a}{b-x-s}ds = -int_{b-x-t}^{b-x}\frac{a}{r}dr = [a\log(r)]_{b-x-t}^{b-x}$ using the substitution $r = b-x-s$. This gives $S_x(t) = e^{-a(\log(b-x)-\log(b-x-t))} = \left(\frac{b-x-t}{b-x}\right)^a$.

   The probability of an individual aged 44 surviving for 10 years is therefore $\left(\frac{b-54}{b-44}\right)^a$. We are therefore given $\left(\frac{b-54}{b-44}\right)^a = 0.982$.

Setting this equal to 0.982 gives $\int_{44}^{54} \frac{a}{b-x} dx = -\log(0.982) = 0.018163971$, or $\frac{(54^2-44^2)}{2}a + 10b = 0.018163971$. $490a + 10b = 0.018163971$.

The life expectancy of an individual aged 65 is

$$\int_0^\infty S_{65}(t)dt = \int_0^{b-65} \left(\frac{b-65-t}{b-65}\right)^a dt$$

We make the substitution $s = \frac{b-65-t}{b-65}$, so $\frac{ds}{dt} = \frac{-1}{b-65}$ to get

$$\mathbb{E}(T_{65}) = \int_0^1 s^a(b-65)ds = \frac{b-65}{a+1}$$

We therefore have

$$\begin{aligned} \frac{b-65}{a+1} &= 19 \\ \left(\frac{b-54}{b-44}\right)^a &= 0.982 \end{aligned}$$

Substituting $b = 19a + 84$ into the second equation gives

$$\left(1 - \frac{10}{19a+40}\right)^a = 0.982$$

. We need to choose the value of a that satisfies this. We try the values given:

	a	$\left(\frac{19a+30}{19a+40}\right)^a$
(i)	0.0443	0.9876
(ii)	0.0654	0.9820
(iii)	0.102	0.9726
(iv)	0.288	0.9310

We therefore see that (ii) $a = 0.0654$ is the correct answer, and the corresponding value of b is $b = 85.24$.

6. *An insurance company prepares the following lifetable for an unhealthy individual.*

x	l_x	d_x
45	*10000.00*	*602.13*
46	*9397.87*	*683.49*
47	*8714.37*	*765.53*
48	*7948.85*	*843.45*
49	*7105.40*	*910.70*
50	*6194.70*	*959.07*
51	*5235.63*	*979.14*
52	*4256.49*	*961.56*
53	*3294.93*	*899.13*
54	*2395.80*	*789.74*
55	*1606.07*	*639.52*

After a medical examination, it determines that the individual's probability of death at age 45 is only 0.0023. The probability of death in each subsequent year remains the same. Prepare a new life table for this individual over the same range using radix 10,000.

Under the new model, the number of lives surviving to age 46 should be 9977, so all numbers in the table should be multiplied by $\frac{9977}{9397.87}$. This gives the following lifetable:

x	l_x	d_x
45	10000.00	23.00
46	9977.00	725.61
47	9251.38	812.70
48	8438.69	895.43
49	7543.26	966.82
50	6576.44	1018.17
51	5558.27	1039.48
52	4518.79	1020.81
53	3497.98	954.54
54	2543.44	838.41
55	1705.04	678.93