ACSC/STAT 3720, Life Contingencies I
Winter 2015
Toby Kenney
Homework Sheet 4
Model Solutions

Basic Questions

1. *Using the lifetable in Table 1, calculate $\ddot{a}_{[35]+3}$ at interest rate $i = 0.06$.*

   The discount rate is given by $d = \frac{1}{1+i} = \frac{0.06}{1.06} = 0.0566$. Using the standard recurrence, starting at $n = 125$, we calculate $A_{[35]+3} = 0.0580485$. We then use the formula $\ddot{a}_{[35]+3} = \frac{1-A_{[35]+3}}{d} = \frac{0.9419515}{0.0566} = 16.64$.

2. *An individual aged 49 for whom Table 1 is appropriate, takes out a 10-year term-insurance policy. The annual premiums are \$7,000, payable at the begining of each year. If the current interest rate is $i = 0.05$, what is the expected present value of the premiums received?*

   Using the standard recurrence, $A_{49:10} = 0.615931$. This gives $\ddot{a}_{49:10} = \frac{1-A_{49:10}}{d}$, where $d = \frac{i}{1+i} = .0476$. We therefore get $\ddot{a}_{49:10} = 8.065$, so the expected present value of the premiums received is $7000 \times 8.05645 = \$56,393$.

3. *An annuity pays out continuously at a rate of \$3,000 a year until the death of a select individual currently aged 67 to whom the lifetable in Table 1 applies. What is the expected present value of this annuity, using the uniform distribution of deaths assumption, and force of interest $\delta = 0.09$?*

   Using the standard recurrence, with $i = e^{0.09} - 1 = 0.094174284$, we calculate $A_{[67]} = 0.128268$. We have $\frac{i}{\delta} = \frac{0.094174284}{0.09} = 1.046387$. Using a UDD assumption, we therefore get $\overline{A}_{[67]} = \frac{i}{\delta}A_{[67]} = 1.046387 \times 0.128268 = 0.13421719$. We therefore get $\overline{a}_{[67]} = \frac{1-\overline{A}_{[67]}}{\delta} = \frac{1-0.13421719}{0.09} = 9.619809$. The annuity pays out at a rate of \$3,000 a year, so the EPV is $3000 \times 9.619809 = \$28,859.43$.

4. *A pension plan pays monthly benefits of \$2500 to an individual aged 68. What is the expected present value of the benefit under the uniform distribution of deaths assumption, interest rate $i^{(12)} = 0.06$ and the lifetable in Table 1?*

   From this we get $i = (1.005)^{12} - 1 = 0.061677812$. Using the standard recursion, this gives $A_{68} = 0.240986$. Using the UDD assumption we have $A_{68}^{(12)} = \frac{0.061677812}{0.06}A_{68} = 0.24772482$. This gives $a_{68}^{(12)} = \frac{1-0.24772482}{0.06} = 12.537919667$, so the EPV of this pension is $2500 \times 12 \times 12.537919667 = \$376,137.59$.

Standard Questions

5. *A pension plan pays an annual benefit of \$17,000 to an individual aged 69, for whom the ultimate part of the lifetable in Table 1 applies. The interest rate is $i = 0.08$, which gives $A_{69} = 0.18358$ and $A_{79} = 0.300473$. The individual wants to change the policy to have guaranteed payments for the first 10 years, but keep the EPV of the benefits the same. What should the new annual payments be?*

The EPV is $17000\ddot{a}_{69} = 17000\frac{1-A_{69}}{d} = 17000\frac{0.81642\times 1.08}{0.08} = 187368.39$. Let R be the new annual payment. If the payments are guaranteed for 10 years, then the individual receives a 10-year annuity due, with present value $R\frac{1.08-(1.08)^{-9}}{0.08} = 7.2468879112R$ and if the individual is still alive at age 79, they receive a life annuity-due, which at that time has EPV $17000\frac{1-A_{79}}{d} = 9.4436145R$. The current EPV of this benefit is $9.4436145{}_{10}p_{65}(1.08)^{-10}R$, since this benefit is 10 years in the future, and is contingent on the individual surviving for 10 years. This is $9.4436145 \times \frac{8549.78}{9386.86} \times (1.08)^{-10}R = 3.9841464561R$. The total EPV of the new benefit is therefore $11.2310343673R$, so equating present values gives $R = \frac{187368.39}{11.2310343673} = 16683.09$.

6. *A man aged 66, to whom the ultimate part of the lifetable in Table 1 applies, wants a pension which will pay $20,000 in a year's time, and thereafter will provide annual pensions increasing by 4% every year (so the second payment when the man turns 68 will be $20,800). What is the expected present value of the benefits of this pension if the current interest rate is $i = 0.11$?*

   Since the pension should increase by 4% each year, the real interest rate is $\frac{0.11-0.04}{1.04} = 0.0673076923$. If the man is still alive in one year's time, the EPV of this pension will be obtained by calculating $\ddot{a}_{67}$ at this real interest rate, and multiplying by $20,000. At this real interest rate, we use the standard recurrence to calculate $A_{67} = 0.207079$. This gives $\ddot{a}_{67} = \frac{(1-0.207079)\times 1.0673076923}{0.0673076923} = 12.5734615727$. The expected benefit of this pension at the time the man is 67 is therefore $12.5734615727 \times 20000 = 251469.23$. This is one year in the future, and is contingent on the man surviving for one year. Therefore, the current EPV is $251469.23p_{66}(1.11)^{-1}$. [Note that we use the numerical interest rate i for this calculation, since the $20,000 annual amount was already adjusted for the first year's inflation.] This EPV is $251469.23 \times \frac{9485.52}{9528.85} \times (1.11)^{-1} = \$225,518.68$.

7. *A woman aged 46 is receiving a pension of $27,000 at the start of each year. She wants to change this to a monthly pension. If the appropriate life table is in Table 1 and the interest rate is $i = 0.04$, then we can calculate $A_{46} = 0.178312$. Use Woolhouse's formula to calculate the monthly pension that has the same expected present value.*

   Woolhouse's formula tells us

$$\ddot{a}^{(12)}_{46} = \ddot{a}_{46} - \frac{12-1}{2\times 12} - \frac{12^2-1}{12\times 12^2}(\delta + \mu_{46})$$

   We have $\delta = \log(1.04) = 0.039220713$. We can approximate μ_{46} as $\frac{q_{45}+q_{46}}{2} = \frac{7.12}{2\times 9930.38} + \frac{7.74}{2\times 9923.26} = .0007484886$. We therefore have

$$\ddot{a}^{(12)}_{46} = \ddot{a}_{46} - \frac{11}{24} - \frac{143}{1728} \times .0399692016 = \ddot{a}_{46} - 0.46164$$

   The expected present value of the annual pension is $27000\ddot{a}_{46}$. If the monthly pension is R every month, then using Woolhouse's formula gives the expected present value of the monthly pension as $12R(\ddot{a}_{46} - 0.46164)$. We therefore have

$$R = \frac{27000\ddot{a}_{46}}{12(\ddot{a}_{46} - 0.46164)}$$

   Substituting the formula $\ddot{a}_{46} = \frac{1-A_{46}}{d}$ gives

$$R = \frac{27000(1-A_{46})}{12(1-A_{46} - 0.46164d)} = 2250\left(1 - 0.0177554218\frac{A_{46}}{1-A_{46}}\right)$$

   We are given that $A_{46} = 0.178312$, so this gives $R = \$2,241.33$.

8. *An individual aged 48 is starting to invest in a pension plan. He wants to receive $26,000 a year, starting at age 65. He plans to pay for this with annual premiums from now until he turns 65 (so the first premium is today, the last premium is on his 64th birthday). The interest rate is* $i = 0.07$. *The insurance company calculates that for this individual,* $A_{65} = 0.178416$ *and* $A_{48} = 0.0716384$. *What should the annual premiums be?*

   From the information we are given, we have $\ddot{a}_{65} = \frac{1-A_{65}}{d} = \frac{0.821584\times 1.07}{0.07} = 12.5584982857$ and $\ddot{a}_{48:17} = \frac{1-A_{48}}{d} - {}_{17}p_{48}(1.07)^{-17}\ddot{a}_{65} = 14.1906701714 - \frac{9568.61}{9907.10} \times (1.07)^{-17} \times 12.5584982857 = 10.3508065758$. We want the EPV of the premiums to be the EPV of the benefits. That is if the premium is P, we want

   $$10.3508065758P = 26000 \times {}_{17}p_{48} \times (1.07)^{-17} \times 12.5584982857$$

   which gives

   $$P = \frac{26000 \times 3.8398635956}{10.3508065758} = \$9,645.28$$

Bonus Question

9. *Consider a policy for a life currently aged* x *which has a death benefit at the end of the year of death of* $s_{\overline{k}|i}$ *if the life dies between ages* $x+k$ *and* $x+k+1$. *What is the expected present value of this benefit?*

   Imagine a fund into which $\frac{1}{1+i}$ is deposited at the start of each year in which the life is alive. The accumulated value in this fund before the payment in the nth year is $s_{\overline{n}|i}$, so this insurance pays the same death benefit as this imaginary fund, which corresponds to an annuity. The EPV of this death benefit is therefore the same as the EPV of the annuity, which is $\frac{\ddot{a}_x}{1+i}$.

Table 1: Select lifetable to be used for questions on this assignment

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
25	9998.75	9997.65	9996.30	9994.66
26	9997.00	9995.83	9994.40	9992.66
27	9995.14	9993.90	9992.38	9990.52
28	9993.16	9991.84	9990.22	9988.24
29	9991.05	9989.65	9987.92	9985.80
30	9988.81	9987.30	9985.46	9983.18
31	9986.40	9984.80	9982.82	9980.38
32	9983.83	9982.11	9979.99	9977.37
33	9981.07	9979.23	9976.95	9974.13
34	9978.11	9976.13	9973.68	9970.64
35	9974.93	9972.79	9970.16	9966.88
36	9971.50	9969.20	9966.36	9962.82
37	9967.80	9965.33	9962.25	9958.44
38	9963.81	9961.14	9957.82	9953.69
39	9959.50	9956.61	9953.02	9948.55
40	9954.84	9951.71	9947.82	9942.98
41	9949.79	9946.41	9942.19	9936.94
42	9944.32	9940.66	9936.08	9930.38
43	9938.39	9934.41	9929.45	9923.26
44	9931.96	9927.64	9922.25	9915.52
45	9924.97	9920.28	9914.42	9907.10
46	9917.37	9912.28	9905.91	9897.94
47	9909.11	9903.58	9896.65	9887.98
48	9900.13	9894.11	9886.57	9877.13
49	9890.36	9883.80	9875.59	9865.30
50	9879.71	9872.57	9863.63	9852.42
51	9868.12	9860.34	9850.59	9838.38
52	9855.48	9847.01	9836.39	9823.08
53	9841.72	9832.48	9820.90	9806.39
54	9826.71	9816.64	9804.02	9788.18
55	9810.34	9799.37	9785.60	9768.33
56	9792.49	9780.52	9765.51	9746.67
57	9773.03	9759.97	9743.60	9723.05
58	9751.79	9737.56	9719.69	9697.28
59	9728.63	9713.10	9693.62	9669.17
60	9703.36	9686.43	9665.17	9638.51
61	9675.80	9657.33	9634.15	9605.07
62	9645.73	9625.59	9600.31	9568.61
63	9612.94	9590.98	9563.42	9528.85
64	9577.18	9553.24	9523.19	9485.52
65	9538.19	9512.09	9479.35	9438.30
66	9495.69	9467.25	9431.58	9386.86
67	9449.37	9418.39	9379.54	9330.85
68	9398.90	9365.17	9322.87	9269.88
69	9343.95	9307.23	9261.20	9203.55
70	9284.12	9244.18	9194.11	9131.43
71	9219.03	9175.59	9121.17	9053.07
72	9148.24	9101.03	9041.91	8967.97
73	9071.30	9020.03	8955.85	8875.63

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
74	8987.73	8932.10	8862.49	8775.52
75	8897.04	8836.71	8761.27	8667.10
76	8798.69	8733.34	8651.66	8549.78
77	8692.13	8621.41	8533.09	8423.00
78	8576.81	8500.36	8404.95	8286.16
79	8452.13	8369.60	8266.68	8138.66
80	8317.52	8228.53	8117.67	7979.93
81	8172.36	8076.57	7957.35	7809.41
82	8016.08	7913.13	7785.15	7626.56
83	7848.11	7737.67	7600.54	7430.89
84	7667.89	7549.66	7403.05	7221.99
85	7474.92	7348.64	7192.27	6999.51
86	7268.77	7134.21	6967.86	6763.22
87	7049.07	6906.07	6729.62	6513.04
88	6815.55	6664.05	6477.46	6249.02
89	6568.09	6408.10	6211.48	5971.42
90	6306.70	6138.35	5931.96	5680.73
91	6031.59	5855.15	5639.41	5377.67
92	5743.19	5559.08	5334.61	5063.27
93	5442.15	5250.97	5018.61	4738.86
94	5129.44	4931.97	4692.79	4406.12
95	4806.33	4603.54	4358.89	4067.08
96	4474.39	4267.51	4018.96	3724.10
97	4135.60	3926.04	3675.44	3379.91
98	3792.25	3581.66	3331.11	3037.57
99	3447.02	3237.23	2989.05	2700.39
100	3102.90	2895.94	2652.63	2371.88
101	2763.19	2561.21	2325.37	2055.64
102	2431.39	2236.61	2010.90	1755.27
103	2111.15	1925.80	1712.81	1474.18
104	1806.12	1632.34	1434.48	1215.44
105	1519.82	1359.55	1178.94	981.65
106	1255.46	1110.36	948.70	774.71
107	1015.81	887.14	745.58	595.71
108	802.96	691.49	570.56	444.87
109	618.23	524.17	423.71	321.41
110	462.04	385.00	304.13	223.65
111	333.80	272.80	210.00	149.10
112	231.99	185.53	138.71	94.62
113	154.19	120.34	87.07	56.74
114	97.30	73.90	51.50	31.84
115	57.78	42.55	28.41	16.52
116	31.92	22.69	14.43	7.81
117	16.15	11.04	6.63	3.30
118	7.34	4.79	2.69	1.21
119	2.90	1.79	0.93	0.37
120	0.95	0.55	0.26	0.09
121	0.23	0.13	0.05	0.01
122	0.03	0.02	0.01	0.00