ACSC/STAT 3720, Life Contingencies I
Winter 2017
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Homework Sheet 7
Model Solutions

Basic Questions

1. *An insurance company sells 1,400 whole life insurance policies with annual net premiums to lives aged 53. The death benefit on these policies is \$500,000. The interest rate is* $i = 0.05$. *In the first year of the policies:*

   - *One policyholder dies.*
   - *The company earns interest* $i = 0.07$.

   The company still uses $i = 0.05$ *as its basis for calculating the policy values. What is the company's annual profit on these policies? [Using the lifetable in Table 1, we have* $A_{53} = 0.166573$ *and* $A_{54} = 0.173724$.*]*

   We calculate $\ddot{a}_{53} = \frac{1.05}{0.05}(1-0.166573) = 17.501967$ and $\ddot{a}_{54} = \frac{1.05}{0.05}(1-0.173724) = 17.351796$. The premium is $\frac{0.166573\times 500000}{17.501967} = \$4758.69$. The policy value after one year is $500000 \times 0.173724 - 4758.69 \times 17.351796 = \$4,290.18$.

   The company receives $4758.69\times 1400 = \$6,662,166$ in premiums, and earns $0.07\times 6662166 = \$466,351.62$ in interest. The company pays out \$500,000 in benefits, and need to keep $1399\times 4290.18 = \$6,001,961.82$ in reserves. The profit on the policies is therefore $6662166 + 466351.62 - 500000 - 6001961.82 = \$626,555.80$.

2. *An insurance company sells 1,600 20-year term insurance policies to lives aged 37 with mortality following the ultimate part of Table 1. The death benefit of these policies is \$600,000. The interest rate is* $i = 0.05$ *and net premiums are payable annually in advance. At this interest rate,* $A_{37} = 0.0827855$ *and* $A_{57} = 0.196764$. *In the first two years of the policy:*

   - *one policyholder dies in each year of the policy.*
   - *The company earns interest* $i = 0.06$ *in the first year of the policy, and* $i = 0.07$ *in the second year.*

   Calculate the asset share of the remaining policies after the second year.

   We calculate $A_{37:\overline{20|}} = A_{37} + {}_{20}p_{57}(1.05)^{-20}(1 - A_{57}) = 0.0827855 + \frac{9788.18}{9970.64}(1.05)^{-20}(1 - 0.196764) = 0.379976802035$ and therefore $\ddot{a}_{37:\overline{20|}} = \frac{1.05}{0.05}(1-0.379976802035) = 13.0204871573$ and $A^{1}_{37:\overline{20|}} = A_{37} - {}_{20}p_{57}(1.05)^{-20}A_{57} = 0.0827855 - 0.196764\frac{9788.18}{9970.64}(1.05)^{-20} = 0.0099842941854$. The premium is therefore $\frac{0.0099842941854\times 600000}{13.0204871573} = \$460.09$.

   In the first year, The company receives $460.09 \times 1600 = \$736,144$ in premiums, with $i = 0.06$, the company has $736144(1.06) = \$780,312.64$ at the end of the year. It now pays a death benefit of \$600,000, so has \$180,312.64 remaining. It now receives $460.09 \times 1599 = \$735,683.91$ in premiums, so it has $180312.64 + 735683.91 = \$915,996.55$. After earning $i = 0.07$, it has $915996.55(1.07) = \$980,116.31$ at the end of the year, and pays another death benefit of \$600,000, leaving \$380,116.31 in assets to be shared among 1,598 policies, meaning the asset share is $\frac{380116.31}{1598} = \$237.87$.

3. *A life aged 44 purchases a whole-life insurance policy with a death benefit of \$900,000. The interest rate is $i = 0.06$. From the lifetable in Table 1, we have $A_{44} = 0.0791340$ and $A_{52} = 0.124241$. Using Woolhouse's formula, this results in a monthly premium of \$499.18*

(a) calculate the policy value after 7 years and 10 months. [You may use the UDD assumption for the distribution of deaths in Year 8, but use Woolhouse's formula to calculate $\ddot{a}_{52}^{(12)}$.]

We calculate $\ddot{a}_{52} = \frac{1.06}{0.06}(1 - 0.124241) = 15.4717423334$ and we get $\delta = \log(1.06) = 0.058268908124$ and $\mu_{52} \approx \frac{1}{2}(q_{51} + q_{52}) = \frac{1}{2}\left(\frac{11.83}{9877.13} + \frac{12.88}{9865.30}\right) = 0.00125165129371$. Woolhouse's formula therefore gives $\ddot{a}_{52}^{(12)} = 15.4717423334 - \frac{11}{24} - \frac{143}{1728}(0.058268908124 + 0.00125165129371) = 15.0084833983$. Working backwards under UDD, we get

$$\begin{aligned}
\ddot{a}_{51\frac{11}{12}}^{(12)} &= \frac{1}{12} + (1.06)^{-\frac{1}{12}}\,{}_{\frac{1}{12}}p_{51\frac{11}{12}}\ddot{a}_{52}^{(12)} \\
&= \frac{1}{12} + (1.06)^{-\frac{1}{12}}\frac{9865.30}{\frac{11}{12}\times 9865.30 + \frac{1}{12}\times 9877.13} \times 15.0084833983 \\
&= 15.0176236796 \\
\ddot{a}_{51\frac{10}{12}}^{(12)} &= \frac{1}{12} + (1.06)^{-\frac{1}{12}}\,{}_{\frac{1}{12}}p_{51\frac{10}{12}}\ddot{a}_{51\frac{11}{12}}^{(12)} \\
&= \frac{1}{12} + (1.06)^{-\frac{1}{12}}\frac{\frac{11}{12}\times 9865.30 + \frac{1}{12}\times 9877.13}{\frac{10}{12}\times 9865.30 + \frac{2}{12}\times 9877.13} \times 15.0176236796 \\
&= 15.0267189259
\end{aligned}$$

This gives $A_{51\frac{10}{12}}^{(12)} = 1 - 12\left(1 - 1.06^{-\frac{1}{12}}\right) \times 15.0267189259 = 0.126531879862$.

The policy value is therefore $0.126531879862 \times 900000 - 15.0267189259 \times 12 \times 499.18 = \$23,866.24$

(b) calculate the policy value after 7 years 9.6 months.

We have ${}_{7\frac{10}{12}}V = 23866.24$, so we have

$$\begin{aligned}
{}_{7+\frac{9.6}{12}}V &= (1.06)^{-\frac{0.4}{12}}\,{}_{\frac{0.4}{12}}p_{7+\frac{9.6}{12}}\,{}_{7\frac{10}{12}}V \\
&= (1.06)^{-\frac{0.4}{12}}\frac{\frac{10}{12}\times 9865.30 + \frac{2}{12}\times 9877.13}{\frac{9.6}{12}\times 9865.30 + \frac{2.4}{12}\times 9877.13} \times 23866.24 \\
&= \$23,818.98
\end{aligned}$$

Standard Questions

4. *An insurance company wants to design a 10-year term policy with continuous premiums so that the policy value is given by ${}_tV = 130t^2(t-10)$. The death benefits at time t are $100000(3+0.1e^{0.04t})$. The policy is sold to a life aged 48, with mortality given by $\mu_x = 0.0000045(1.12)^x$. Calculate the premiums as a function of time if force of interest is $\delta = 0.043$.*

Thiel's differential equation gives

$$\begin{aligned}
\frac{d}{dt}({}_tV) &= \delta_t V + P_t - \mu_{x+t}(B_t - {}_tV) \\
\frac{d}{dt}(130t^2(t-10)) &= 0.043 \times 130t^2(t-10) + P_t - 0.0000045(1.12)^{x+t}(100000(3+0.1e^{0.04t}) - 130t^2(t-10)) \\
t^2(390t-2600) &= 0.043 \times 130t^2(t-10) + P_t - 0.0000045(1.12)^{x+t}(100000(3+0.1e^{0.04t}) - 130t^2(t-10)) \\
P_t &= t^2(390t-2600) - 0.043 \times 130t^2(t-10) \\
&\quad + 0.0000045(1.12)^{x+t}(100000(3+0.1e^{0.04t}) - 130t^2(t-10)) \\
P_t &= 130t^2(2.957t - 19.57) + 0.0000045(1.12)^{x+t}(100000(3+0.1e^{0.04t}) - 130t^2(t-10))
\end{aligned}$$

5. *An insurance company is valuing its policies. It finds that the total value last year of a large group of 200 policies with a variety of premiums and benefits was \$16,000,000. The total annual premium for all these policies is \$1,800,000. The interest rate is* $i = 0.05$. *The expected number of policyholders who die is 4.7, the expected total benefits paid is \$1,980,000 and the expected total policyvalue (at the end of the year) of all policyholders who die is \$230,000. There are no expenses associated with the policies. During the year 2 policyholders die. The first had mortality* $q_x = 0.009$, *premium \$31,000, received benefits of \$600,000 and started the year with policy value \$29,000; the second had mortality* $q_x = 0.017$, *premium \$24,500 received benefits of \$500,000 and started the year with policy value \$134,000. What is the total value of the remaining 198 policies in this group?*

   The recurrence for annual policy values is

$$ {}_tV = (1.05)^{-1}(p_{x+t\,t+1}V + q_{x+t}B) - P $$

   Adding these recurrences for the portfolio, we have

$$\begin{aligned}
\sum_t V &= (1.05)^{-1}\left(\sum p_{x+t\,t+1}V + \sum q_{x+t}B\right) - \sum P \\
16000000 &= (1.05)^{-1}\left(\sum_{t+1} V + \sum q_{x+t}(B - {}_{t+1}V)\right) - 1800000 \\
17800000 &= (1.05)^{-1}\left(\sum_{t+1} V + 1980000 - 230,000)\right) \\
\sum_{t+1} V &= 1.05 \times 17800000 - 1980000 + 230000 \\
\sum_{t+1} V &= 16940000
\end{aligned}$$

For the two policies where the policyholder died, the end of year policy values were

$$\begin{aligned}
{}_{t+1}V &= (1.05) \times (29000 + 31000) - 0.009 \times (600000 - {}_{t+1}V) \\
0.992({}_{t+1}V) &= 63000 - 5400 \\
{}_{t+1}V &= \$58,064.516129 \\
{}_{t+1}V &= (1.05) \times (134000 + 25400) - 0.017 \times (500000 - {}_{t+1}V) \\
0.983({}_{t+1}V) &= 158870 \\
{}_{t+1}V &= \$161,617.497457
\end{aligned}$$

The total policy value of the remaining policies is therefore

$$\sum_{t+1} V = 16940000 - 58064.516129 - 161617.497457 = \$16,720,317.99$$

Table 1: Select lifetable to be used for questions on this assignment

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
25	9998.75	9997.65	9996.30	9994.66	74	8987.73	8932.10	8862.49	8775.52
26	9997.00	9995.83	9994.40	9992.66	75	8897.04	8836.71	8761.27	8667.10
27	9995.14	9993.90	9992.38	9990.52	76	8798.69	8733.34	8651.66	8549.78
28	9993.16	9991.84	9990.22	9988.24	77	8692.13	8621.41	8533.09	8423.00
29	9991.05	9989.65	9987.92	9985.80	78	8576.81	8500.36	8404.95	8286.16
30	9988.81	9987.30	9985.46	9983.18	79	8452.13	8369.60	8266.68	8138.66
31	9986.40	9984.80	9982.82	9980.38	80	8317.52	8228.53	8117.67	7979.93
32	9983.83	9982.11	9979.99	9977.37	81	8172.36	8076.57	7957.35	7809.41
33	9981.07	9979.23	9976.95	9974.13	82	8016.08	7913.13	7785.15	7626.56
34	9978.11	9976.13	9973.68	9970.64	83	7848.11	7737.67	7600.54	7430.89
35	9974.93	9972.79	9970.16	9966.88	84	7667.89	7549.66	7403.05	7221.99
36	9971.50	9969.20	9966.36	9962.82	85	7474.92	7348.64	7192.27	6999.51
37	9967.80	9965.33	9962.25	9958.44	86	7268.77	7134.21	6967.86	6763.22
38	9963.81	9961.14	9957.82	9953.69	87	7049.07	6906.07	6729.62	6513.04
39	9959.50	9956.61	9953.02	9948.55	88	6815.55	6664.05	6477.46	6249.02
40	9954.84	9951.71	9947.82	9942.98	89	6568.09	6408.10	6211.48	5971.42
41	9949.79	9946.41	9942.19	9936.94	90	6306.70	6138.35	5931.96	5680.73
42	9944.32	9940.66	9936.08	9930.38	91	6031.59	5855.15	5639.41	5377.67
43	9938.39	9934.41	9929.45	9923.26	92	5743.19	5559.08	5334.61	5063.27
44	9931.96	9927.64	9922.25	9915.52	93	5442.15	5250.97	5018.61	4738.86
45	9924.97	9920.28	9914.42	9907.10	94	5129.44	4931.97	4692.79	4406.12
46	9917.37	9912.28	9905.91	9897.94	95	4806.33	4603.54	4358.89	4067.08
47	9909.11	9903.58	9896.65	9887.98	96	4474.39	4267.51	4018.96	3724.10
48	9900.13	9894.11	9886.57	9877.13	97	4135.60	3926.04	3675.44	3379.91
49	9890.36	9883.80	9875.59	9865.30	98	3792.25	3581.66	3331.11	3037.57
50	9879.71	9872.57	9863.63	9852.42	99	3447.02	3237.23	2989.05	2700.39
51	9868.12	9860.34	9850.59	9838.38	100	3102.90	2895.94	2652.63	2371.88
52	9855.48	9847.01	9836.39	9823.08	101	2763.19	2561.21	2325.37	2055.64
53	9841.72	9832.48	9820.90	9806.39	102	2431.39	2236.61	2010.90	1755.27
54	9826.71	9816.64	9804.02	9788.18	103	2111.15	1925.80	1712.81	1474.18
55	9810.34	9799.37	9785.60	9768.33	104	1806.12	1632.34	1434.48	1215.44
56	9792.49	9780.52	9765.51	9746.67	105	1519.82	1359.55	1178.94	981.65
57	9773.03	9759.97	9743.60	9723.05	106	1255.46	1110.36	948.70	774.71
58	9751.79	9737.56	9719.69	9697.28	107	1015.81	887.14	745.58	595.71
59	9728.63	9713.10	9693.62	9669.17	108	802.96	691.49	570.56	444.87
60	9703.36	9686.43	9665.17	9638.51	109	618.23	524.17	423.71	321.41
61	9675.80	9657.33	9634.15	9605.07	110	462.04	385.00	304.13	223.65
62	9645.73	9625.59	9600.31	9568.61	111	333.80	272.80	210.00	149.10
63	9612.94	9590.98	9563.42	9528.85	112	231.99	185.53	138.71	94.62
64	9577.18	9553.24	9523.19	9485.52	113	154.19	120.34	87.07	56.74
65	9538.19	9512.09	9479.35	9438.30	114	97.30	73.90	51.50	31.84
66	9495.69	9467.25	9431.58	9386.86	115	57.78	42.55	28.41	16.52
67	9449.37	9418.39	9379.54	9330.85	116	31.92	22.69	14.43	7.81
68	9398.90	9365.17	9322.87	9269.88	117	16.15	11.04	6.63	3.30
69	9343.95	9307.23	9261.20	9203.55	118	7.34	4.79	2.69	1.21
70	9284.12	9244.18	9194.11	9131.43	119	2.90	1.79	0.93	0.37
71	9219.03	9175.59	9121.17	9053.07	120	0.95	0.55	0.26	0.09
72	9148.24	9101.03	9041.91	8967.97	121	0.23	0.13	0.05	0.01
73	9071.30	9020.03	8955.85	8875.63	122	0.03	0.02	0.01	0.00