MATH/STAT 4703, Actuarial Models II

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Formula Sheet

General Mathematics

- Quadratic Formula: Solution to $ax^2+bx+c=0$ is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- Gamma function: $\Gamma(\alpha)=\int_0^\infty x^\alpha e^{-x}dx$ satisfies $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$.

Moments

Centralised moments in terms of uncentralised moments:

$$\begin{aligned}
\mu_2 &= \mu_2' - \mu^2\\
\mu_3 &= \mu_3' - 3\mu\mu_2' + 2\mu^3\\
\mu_4 &= \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4
\end{aligned}$$

Risk Measures

- Standard deviation principle $r=\mu+a\sigma$.
- Value at Risk $r=\pi_p$.
- Tail Value at Risk $$\begin{aligned} r &= \frac{\int_{\pi_p}^\infty xf(x)dx}{1-p}\\ &= \pi_p + \frac{\int_{\pi_p}^\infty S(x)dx}{1-p}\end{aligned}$$

Continuous Distributions: Transformed Beta family

Transformed Beta

Inverse of Transformed Beta with $\alpha=\tau$, $\tau=\alpha$, $\theta=\frac{1}{\theta}$.

Density function	$f(x)=\left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right)\frac{\gamma\left(\frac{x}{\theta}\right)^{\tau\gamma}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+\tau}}$
Mean	$\theta\frac{\Gamma\left(\tau+\frac{1}{\gamma}\right)\Gamma\left(\alpha-\frac{1}{\gamma}\right)}{\Gamma(\tau)\Gamma(\alpha)}$
Raw Moments	$\mu_k'=\theta^k\frac{\Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\tau)\Gamma(\alpha)}$
Moment Generating Function	Undefined

0.1 Burr

Transformed Beta with $\tau=1$.

Density function	$f(x)=\frac{\alpha\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$
Mean	$\theta\frac{\Gamma\left(\alpha-\frac{1}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}\right)}{\Gamma(\alpha)}$
Raw Moments	$\mu_n'=\theta^n\frac{n\Gamma\left(\alpha-\frac{n}{\gamma}\right)\Gamma\left(\frac{n}{\gamma}\right)}{\Gamma(\alpha)}$
Moment Generating Function	Undefined

Inverse Burr

Transformed Beta with $\alpha=1$.

Density function	$f(x)=\frac{\tau\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\tau+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$
Mean	$\theta\frac{\Gamma\left(\tau+\frac{1}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}\right)}{\Gamma(\tau)}$
Raw Moments	$\mu_k'=\theta^k\frac{\Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)}{\Gamma(\tau)}$
Moment Generating Function	Undefined

Generalised Pareto

Transformed Beta with $\gamma=1$.

Density function	$f(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\left(\frac{x}{\theta}\right)^{\tau}}{x\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha+\tau}}$
Mean	$\theta\frac{\tau}{\alpha-1}$
Raw Moments	$\mu_k' = \theta^k \frac{\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\tau)\Gamma(\alpha)}$
Moment Generating Function	Undefined

Pareto

Transformed Beta with $\tau = \gamma = 1$.

Density function	$f(x) = \frac{\alpha}{\theta\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha}}$
Mean	$\frac{\theta}{\alpha-1}$ (if $\alpha > 1$)
Variance	$\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$ (if $\alpha > 2$)
Raw Moments	$\mu_k' = \theta^k \frac{\Gamma(1+k)\Gamma(\alpha-k)}{\Gamma(\alpha)}$
Moment Generating Function	Undefined

Inverse Pareto

Transformed Beta with $\alpha = \gamma = 1$.

Density function	$f(x) = \frac{\tau\left(\frac{\theta}{x}\right)}{x\left(1+\left(\frac{\theta}{x}\right)\right)^{\tau+1}}$
Survival Function	$1 - \frac{1}{\left(1+\left(\frac{\theta}{x}\right)\right)^{\tau}}$
Mean	undefined
Moment Generating Function	Undefined

log-logistic

Transformed Beta with $\alpha = \tau = 1$.

Density function	$f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^{\gamma}}{x\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{2}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)}$
Mean	$\theta\Gamma\left(1+\frac{1}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}\right)$
Raw Moments	$\mu_k' = \theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)$
Moment Generating Function	Undefined

Paralogistic

Transformed Beta with $\tau = 1$, $\alpha = \gamma$.

Density function	$f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^{\gamma}}{x\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\gamma+1}}$
Survival function	$S(x) = \frac{1}{\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\gamma}}$
Mean	$\theta\frac{\Gamma\left(\gamma-\frac{1}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}\right)}{\Gamma(\gamma)}$
Raw Moments	$\mu_k' = \theta^k \frac{\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(\gamma-\frac{k}{\gamma}\right)}{\Gamma(\gamma)}$
Variance	
Moment Generating Function	Undefined

Inverse Paralogistic

Transformed Beta with $\alpha = 1$, $\tau = \gamma$.

Density function	$f(x) = \frac{\gamma\left(\frac{\theta}{x}\right)^{\gamma}}{x\left(1+\left(\frac{\theta}{x}\right)^{\gamma}\right)^{\gamma+1}}$
Survival function	$S(x) = 1 - \frac{1}{\left(1+\left(\frac{\theta}{x}\right)^{\gamma}\right)^{\gamma}}$
Mean	
Raw Moments	$\mu'_k = \theta^k \frac{\Gamma\left(\gamma+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)}{\Gamma(\gamma)}$
Variance	
Excess loss	
Moment Generating Function	Undefined

Continuous Distributions: Transformed Gamma family

Transformed Gamma

Limit of Transformed Beta as $\alpha \to \infty$ and $\theta \to \infty$ with $\alpha\theta^{\alpha} = \xi$.

Density function	$f(x) = \frac{\tau\left(\frac{x}{\theta}\right)^{\tau\alpha} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x\Gamma(\alpha)}$
Mean	$\mu = \theta\frac{\tau\left(\alpha+\frac{1}{\tau}\right)}{\tau(\alpha)}$
Raw moments	$\mu'_n = \theta^n \frac{\Gamma\left(\alpha+\frac{n}{\tau}\right)}{\Gamma(\alpha)}$

Gamma

Transformed Gamma with $\tau = 1$

Density function	$f(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} e^{-\left(\frac{x}{\theta}\right)}}{x\Gamma(\alpha)}$
Survival function (for $\alpha \in \mathbb{Z}^+$)	$S(x) = e^{-\frac{x}{\theta}}(1 + \cdots + \frac{\left(\frac{x}{\theta}\right)^{\alpha-1}}{(\alpha-1)!})$
Mean	$\mu = \theta\alpha$
Raw moments	$\mu'_n = \theta^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$
Variance	$\mu_n = \theta^n \alpha$
Moment Generating Function	$M(t) = \frac{1}{(1-\theta t)^{\alpha}}$

Weibull

Transformed Gamma with $\alpha = 1$

Density function	$f(x) = \frac{\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x}$
Survival function	$e^{-\left(\frac{x}{\theta}\right)^{\tau}}$
Mean	$\mu = \theta\Gamma\left(1+\frac{1}{\tau}\right)$
Raw moments	$\mu'_n = \theta^n\Gamma\left(1+\frac{n}{\tau}\right)$

Exponential

Transformed Gamma with $\alpha = \tau = 1$

Density function	$f(x) = \frac{e^{-\left(\frac{x}{\theta}\right)}}{\theta}$
Survival function	$e^{-\frac{x}{\theta}}$
Mean	$\mu = \theta$
Raw moments	$\mu'_n = n!\theta^n$
Variance	$\mu_n = \theta^n$
Excess loss	$\theta e^{-\frac{x}{\theta}}$
Moment Generating Function	$M(t) = \frac{1}{1-\theta t}$

Inverse Transformed Gamma

Inverse of transformed gamma with $\theta = \frac{1}{\theta}$.

Density function	$f(x) = \frac{\tau\left(\frac{\theta}{x}\right)^{\tau\alpha} e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x\Gamma(\alpha)}$
Mean	$\mu = \theta\frac{\Gamma\left(\alpha-\frac{1}{\tau}\right)}{\Gamma(\alpha)}$ (if $\tau\alpha > 1$)
Raw moments	$\mu'_n = \theta^n\frac{\Gamma\left(\alpha-\frac{n}{\tau}\right)}{\Gamma(\alpha)}$ (if $\tau\alpha > n$)

Inverse Gamma

Inverse Transformed Gamma with $\tau = 1$. Inverse of gamma distribution with $\theta = \frac{1}{\theta}$.

Density function	$f(x) = \frac{\left(\frac{\theta}{x}\right)^{\alpha} e^{-\left(\frac{\theta}{x}\right)}}{x\Gamma(\alpha)}$
Survival function (for $\alpha \in \mathbb{Z}^+$)	$S(x) = 1 - e^{-\frac{\theta}{x}}(1 + \cdots + \frac{\left(\frac{\theta}{x}\right)^{\alpha-1}}{(\alpha-1)!})$
Mean	$\mu = \frac{\theta}{\alpha-1}$ (if $\alpha > 1$)
Raw moments	$\mu'_n = \theta^n\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}$ (if $\alpha > n$)
Variance	$\mu_2 = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$

Inverse Weibull

Inverse Transformed Gamma with $\alpha = 1$. Inverse of Weibull distribution with $\theta = \frac{1}{\theta}$.

Density function	$f(x) = \frac{\tau\left(\frac{\theta}{x}\right)^{\tau} e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x}$
Survival function	$1 - e^{-\left(\frac{\theta}{x}\right)^{\tau}}$
Mean	$\mu = \theta\Gamma\left(1 - \frac{1}{\tau}\right)$ (if $\tau > 1$)
Raw moments	$\mu'_n = \theta^n\Gamma\left(1 - \frac{n}{\tau}\right)$ (if $\tau > n$)
Moment Generating Function	Undefined

Inverse Exponential

Inverse Transformed Gamma with $\tau = \alpha = 1$, inverse of exponential with $\theta = \frac{1}{\theta}$.

Density function	$f(x) = \frac{\theta e^{-\left(\frac{\theta}{x}\right)}}{x^2}$
Survival function	$1 - e^{-\frac{\theta}{x}}$
Mean	Undefined

Linear Exponential Family

Density	$f_\theta(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$
mean	$\mu(\theta) = \frac{q'(\theta)}{q(\theta)r'(\theta)}$
Variance	$\mu_2(\theta) = \frac{\mu'(\theta)}{r'(\theta)}$

Normal

Density function	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Mean	$\mu = \mu$
Variance	σ^2
Moment Generating Function	$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Uniform

Density function	$f(x) = \frac{1}{b-a}$ (for $a < x < b$)
Survival funtion	$S(x) = \frac{b-x}{b-a}$ (for $a \leqslant x \leqslant b$)
Mean	$\mu = \frac{a+b}{2}$
Variance	$\frac{(b-a)^2}{12}$
Moment Generating Function	$M(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$

Discrete Distributions

Binomial

Probability	$p_k = \binom{n}{k} p^k (1-p)^{n-k}$
mean	$\mu = np$
raw moments	$\mathbb{E}(X \cdots (X+1-m)) = n \cdots (n+1-m)p^m$
Variance	$\mu_2 = np(1-p)$
p.g.f.	$P(z) = (1-p+pz)^n$
$(a,b,0)$-class	$a = -\frac{p}{1-p}$, $b = \frac{(n-1)p}{1-p}$
zero-truncated probability	$p_1^T = \frac{np(1-p)^{n-1}}{1-(1-p)^n}$

Poisson

Limit of binomial as $n \to \infty$, $p \to 0$ with $np = \lambda$.

Probability	$p_k = e^{-\lambda}\frac{\lambda^k}{k!}$
mean	$\mu = \lambda$
raw moments	$\mathbb{E}(X(X-1)\cdots(X+1-m)) = \lambda^m$
Variance	$\mu_k = \lambda$
p.g.f.	$P(z) = e^{\lambda(z-1)}$
$(a,b,0)$-class	$a = 0$, $b = \lambda$
zero-truncated probability	$p_1^T = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$

Negative Binomial

- Gamma mixture of Poisson distributions where λ follows a gamma distribution with $\theta = \beta$ and $\alpha = r$.
- Number of successes before r failures if probability of success is $\frac{\beta}{1+\beta}$.
- Compound Poisson-Logarithmic distribution, where $\lambda = r \log\left(\frac{1}{1+\beta}\right)$ and $a = \frac{\beta}{1+\beta}$.

Probability	$p_k = \binom{k+r-1}{k}\left(\frac{\beta}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^r$
	$= \frac{r(r+1)\cdots(r+k-1)}{k!}\left(\frac{\beta}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^r$
mean	$\mu = r\beta$
Variance	$\mu_n = r\beta(1+\beta)\cdots(n-1+\beta)$
p.g.f.	$P(z) = \left(\frac{1}{1+\beta-\beta z}\right)^r$
$(a,b,0)$-class	$a = \frac{\beta}{1+\beta}$, $b = \frac{(r-1)\beta}{1+\beta}$
zero-truncated probability	$p_1^T = \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)}$

$(a,b,0)$ and $(a,b,1)$ Classes

$p_k = \left(a + \frac{b}{k}\right) p_{k-1}$ for $k > 1$ (and for $k > 0$ in the $(a,b,0)$ class).

mean	$\mu = \frac{a+b}{1-a}$
Variance	$\mu_2 = \frac{a+b}{(1-a)^2}$
p.g.f.	$P(z) = \left(\frac{1-az}{1-a}\right)^{-\left(1+\frac{b}{a}\right)}$
zero-truncated mean	$\mu = \frac{a+b}{(1-a)\left(1-(a+b)^{1+\frac{b}{a}}\right)}$
zero-truncated probability	$p_1^T = \frac{a+b}{(1-a)^{-\left(1+\frac{b}{a}\right)}-1}$

Logarithmic distribution

Negative binomial with $r = 0$. $(a,b,1)$-class with $a+b=0$, $a = \frac{\beta}{1+\beta}$.

zero-truncated probability	$p_1^T = \frac{-a}{\log(1-a)} = \frac{\beta}{(1+\beta)\log(1+\beta)}$
probability	$p_n = \frac{a^{n-1}}{n} p_1$
mean	$\mu = \frac{-a}{(1-a)(\log(1-a))} = \frac{\beta}{\log(1+\beta)}$
Variance	$\mu_2 = \frac{p_1 - p_1^2}{(1-a)^2} = \frac{\beta(1+\beta)}{\log(1+\beta)} - \frac{\beta}{\log(1+\beta)^2}$
p.g.f.	$P(z) = -\frac{\log(1-az)}{\log(1-a)}$
$(a,b,1)$-class	$a = \frac{\beta}{1+\beta}$, $b = -\frac{\beta}{1+\beta}$

Compound Distributions

Moments:

Let the moments of the primary distribution be $\mu, \mu_2, \mu_3, \ldots$, and the moments of the secondary distribution by $\nu, \nu_2, \nu_3, \ldots$. The moments of the compound distribution are given by:

$\mu\nu$
$\mu\nu_2 + \mu_2\nu^2$
$\mu\nu_3 + \mu_2\nu\nu_2 + \mu_3\nu^3$.

Recursive formula:

If the primary distribution is a member of the $(a,b,1)$-class, the probability mass function is defined as

$$f_S(k) = \frac{(p_1 - (a+b)p_0)f_X(k) + \sum_{i=1}^{k}\left(a + \frac{bi}{k}\right) f_X(i) f_S(k-i)}{1 - af_X(0)}$$

where:

- f_X is the probability mass function of the secondary distribution
- f_S is the probability mass function of the compound distribution
- p_n is the probability that the primary distribution is n (so $p_n = \left(a + \frac{b}{n}\right) p_{n-1}$)

Non-parametric Estimators

Greenwood's formula

$$\mathrm{Var}(S_n(y_j)) \approx S_n(y_j)^2 \sum_{i=1}^{j} \frac{s_i}{r_i(r_i - s_i)}$$

where

- y_i is the ith observed data point in increasing order.
- s_i is the frequency of the observation y_i
- r_i is the size of the risk set at observation y_i.

Log-transformed Confidence intervals

$[S_n(x)^{\frac{1}{U}}, S_n(X)^U]$, where $U = e^{\Phi^{-1}\left(\frac{\alpha}{2}\right)\frac{\sigma}{S_n(x)\log(S_n(x))}}$.

- α is the confidence level (so for a 95% confidence interval, $\alpha = 0.05$).
- σ is the standard deviation of the estimator $S_n(x)$.

Hypothesis Tests

Anderson-Darling test

- Test statistic $n\int_t^u \frac{(F_n(x)-F^*(x))^2}{F^*(x)(1-F^*(x))} f^*(x)\,dx$
- For complete data, given by the formula:

$$-nF^*(u) + n\sum_{i=1}^{k}(F_n(y_i))^2\left(\log(F^*(y_i+1)) - \log(F^*(y_i)\right)$$
$$+n\sum_{i=0}^{k}(1-F_n(y_i))^2\left(\log(1-F^*(y_i)) - \log(1-F^*(y_{i+1})\right)$$

where

- n is sample size.
- Unique observed values are $t = y_0 < y_1 < \cdots < y_k < y_{k+1} = u$
- t is the (left) truncation point (can be $-\infty$ or 0 if no truncation).
- u is the (right) censorship point (can be ∞ if no censorship).

Credibility Theory

Bühlmann Credibility

- $Z = \frac{n}{n+\frac{v}{a}}$ where:
  - n is the sample size (or total exposure)
  - v is the expected process variance.
  - a is the variance of hypothetical means.

Simulation

$(a, b, 0)$ Distributions

- Simulate exponential distributions $T_0, \ldots, T_k$ with rates λ_k (mean $\frac{1}{\lambda_k}$) until the sum exceeds 1.
- Simulated value is first k such that $T_0 + \cdots + T_k > 1$.
- Values of λ_k are $c + dk$ where c and d are:

Distribution	c	d
Poisson	λ	0
Binomial	$-n\log(1-p)$	$\log(1-p)$
Negative Binomial	$r\log(1+\beta)$	$\log(1+\beta)$

Normal Distribution

- Box-Muller transform

$$Z_1 = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$$
$$Z_2 = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$$

- Polar method

$$
\begin{aligned}
X_1 &= 2U_1 - 1 \\
X_2 &= 2U_2 - 1 \\
W &= {X_1}^2 + {X_2}^2 \\
Y &= \sqrt{\frac{-2\log(W)}{W}} \\
Z_1 &= YX_1 \\
Z_2 &= YX_2
\end{aligned}
$$

Discard if $W > 1$.