ACSC/STAT 4720, Life Contingencies II
Fall 2015
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Homework Sheet 3
Model Solutions

Basic Questions

1. *An individual aged 37 has a current salary of \$116,000. The salary scale is $s_y = 1.04^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 67.*

   The final average salary is

   $$\frac{1}{3} \times 116000(1.04^{30} + 1.04^{29} + 1.04^{28}) = \$361949.09$$

   [The meaning of current salary could be interpreted as salary for the coming year in which case the answer would be lower by a factor of 1.04, giving \$348027.97.]

2. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 70% of final average salary for an employee who enters the plan at exact age 35, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 40% of the life annuity.*
   - *At age 65, the employee is married to someone aged 60.*
   - *The salary scale is $s_y = 1.05^y$.*
   - *Mortalities are independent and given by $\mu_x = 0.0000016(1.087)^x$.*
   - *A fixed percentage of salary is payable monthly in arrear.*
   - *Contributions earn an annual rate of 7%.*
   - *The value of the life annuity is based on $i = 0.045$.*

   Calculate the percentage of salary payable monthly to achieve the target replacement rate under these assumptions.

   Let the employee's salary last year be S. At age 65, the employee's final average salary is $S\frac{(1.05^{30}+1.05^{29}+1.05^{28})}{3} = 4.119402S$.

   The employee is therefore assumed to purchase a continuous life annuity at an annual rate of $0.7\times 4.119402S = 2.883581S$, plus a reversionary annuity at an annual rate of $0.4 \times 2.883581S = 1.153433S$.

   We have

$$\begin{aligned}
\overline{a}_{65} &= \int_0^\infty e^{-0.045t} e^{-0.0000016(1.087)^{65}\frac{(1.087)^t-1}{\log(1.087)}}\,dt \\
&= \int_0^\infty e^{-0.045t} e^{-0.0000016(1.087)^{65}\frac{(1.087)^t-1}{\log(1.087)}}\,dt \\
&= 20.15021
\end{aligned}$$

The value of the reversionary annuity is

$$\begin{aligned}
\overline{a}_{65|60} &= \int_0^\infty e^{-0.045t}(1-e^{-0.0000016(1.087)^{65}\frac{(1.087)^t-1}{\log(1.087)}})e^{-0.0000016(1.087)^{60}\frac{(1.087)^t-1}{\log(1.087)}}\,dt \\
&= 0.9766791
\end{aligned}$$

The total EPV of the pensions if the employee survives to age 65 is therefore $20.15021\times 2.883581S+0.9766791\times 1.153433S = 59.23131S$.

Let x be the employee's last months salary. If the employee invests all monthly salary in the pension plan, the final amount of money accumulated would be

$$\begin{aligned}
x\sum_{i=1}^{30\times 12}(1.05)^{\frac{i}{12}}(1.07)^{30-\frac{i}{12}} &= (1.07)^{30}\sum_{i=1}^{30\times 12}\left(\frac{1.05}{1.07}\right)^{\frac{i}{12}} \\
&= \left(\frac{1.05}{1.07}\right)^{\frac{1}{12}}(1.07)^{30}\frac{1-\left(\frac{1.05}{1.07}\right)^{30}}{1-\left(\frac{1.05}{1.07}\right)^{\frac{1}{12}}} \\
&= 1.05^{\frac{1}{12}}x\frac{1.07^{30}-1.05^{30}}{1.07^{\frac{1}{12}}-1.05^{\frac{1}{12}}} \\
&= 2090.932x
\end{aligned}$$

We can calculate $S = x(1+(1.05)^{-\frac{1}{12}}+\cdots+(1.05)^{-\frac{11}{12}}) = 11.68817x$, so the accumulated values of all salary invested monthly is $\frac{2090.932}{11.68817}S = 178.893S$.

The percentage of monthly salary needed is therefore $\frac{59.23131}{178.893} = 0.3310991$, or 33.11%.

3. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
30	1.000000	39	1.350398	48	1.845766	57	2.553877
31	1.033333	40	1.397268	49	1.912422	58	2.649694
32	1.067933	41	1.445983	50	1.981785	59	2.749515
33	1.103853	42	1.496620	51	2.053975	60	2.853522
34	1.141149	43	1.549263	52	2.129115	61	2.961903
35	1.179879	44	1.604000	53	2.207337	62	3.074855
36	1.220103	45	1.660921	54	2.288777	63	3.192585
37	1.261887	46	1.720122	55	2.373580	64	3.315310
38	1.305295	47	1.781702	56	2.461894	65	3.443256

An employee aged 51 and 4 months has 14 years of service, and a current salary of \$96,000 (for the coming year). She has a defined benefit pension plan with $\alpha = 0.015$ and S_{Fin} is the average of her last 3 years' salary. The employee's mortality is given by $\mu_x = 0.00000195(1.102)^x$. The pension benefit is payable monthly in advance. The interest rate is $i = 0.03$. Calculate the EPV of the accrued benefit under the assumption that the employee retires at age 65.

We use linear interpolation to estimate $s_{51.3333333333} = \frac{s_{52}+2s_{51}}{3} = \frac{2\times 2.053975+2.129115}{3} = 2.079022$. If the employee retires at age 65, her final average salary is therefore $96000\frac{3.074855+3.192585+3.315310}{3\times 2.079022} = \$147496.30$. With $\alpha = 0.015$ and 14 years of service, the annual rate of payment is $0.21 \times 147496.30 = \$30,974.22$. The probability that the employee is alive at age x is $e^{-0.00000195\frac{(1.102)^x-(1.102)^{51.3333333333}}{\log(1.102)}}$. The EPV of the accrued benefit is therefore

$$\frac{30974.22}{12}\sum_{i=0}^{\infty} e^{-0.00000195\frac{(1.102)^{65+\frac{i}{12}}-(1.102)^{51.3333333333}}{\log(1.102)}}(1.03)^{-13-\frac{2}{3}-\frac{i}{12}} = \$473,333.07$$

Standard Questions

4. *An employee aged 38 has been working with a company for 14 years. The employee's salary last year was \$62,000. The salary scale is the same as for Question 3. The service table is given below:*

t	${}_tp^{(00)}$	1	2	3
0	*10000.00*	*118.76*	*0*	*0.51*
1	*9880.73*	*112.29*	*0*	*0.58*
2	*9767.86*	*107.16*	*0*	*0.65*
3	*9660.05*	*101.84*	*0*	*0.73*
4	*9557.49*	*96.80*	*0*	*0.82*
5	*9459.86*	*92.02*	*0*	*0.93*
6	*9366.91*	*87.50*	*0*	*1.04*
7	*9278.37*	*83.19*	*0*	*1.18*
8	*9193.99*	*80.11*	*0*	*1.32*
9	*9112.57*	*75.21*	*0*	*1.49*
10	*9035.87*	*71.48*	*0*	*1.68*
11	*8962.71*	*67.92*	*0*	*1.89*
12	*8892.90*	*64.51*	*0*	*2.12*
13	*8826.26*	*61.23*	*0*	*2.39*
14	*8762.64*	*58.07*	*0*	*2.69*
15	*8701.88*	*55.03*	*0*	*3.03*

t	${}_tp^{(00)}$	1	2	3
16	*8643.83*	*52.06*	*0*	*3.41*
17	*8588.36*	*49.18*	*0*	*3.84*
18	*8535.34*	*46.37*	*0*	*4.32*
19	*8484.64*	*43.62*	*0*	*4.86*
10	*8436.17*	*40.90*	*0*	*5.47*
21	*8389.80*	*38.21*	*0*	*6.15*
22^-	*8345.44*		*959.64*	
22	*7385.80*	*21.70*	*119.91*	*5.79*
23	*7238.40*	*18.30*	*108.44*	*6.38*
24^-	*7105.28*		*1203.54*	
24	*5901.74*	*10.81*	*384.29*	*5.86*
25	*5500.78*	*9.14*	*639.20*	*6.15*
26	*4846.29*	*7.73*	*351.32*	*6.10*
27^-	*4481.14*		*4481.14*	

Mortality follows a Gompertz model with $B = 0.00000127$ *and* $C = 1.094$. *The member's current salary is $92,000. If the member withdraws before age 60, he receives a defered pension starting from age 65, with 2% COLA. The death benefit of the plan is three times the employee's final average salary if the employee is still working at the time of death. If the employee has withdrawn, the death benefit is three times final average salary with COLA of 2%. The accrual rate for the pension is 0.02. Pension payments are made annually in advance. The interest rate is* $i = 0.06$.

Calculate the EPV of the accrued benefit. [You may assume that events happen in the middle of each year.]

At an accrual rate of 0.02, the accrued benefit is 26% of final average salary. If the employee retires at age x with a final average salary S (after applying COLA), then the EPV of the pension is given by

$$S\sum_{i=0}^{\infty} e^{-\frac{0.00000127}{\log(1.094)}((1.094)^{x+i}-(1.094)^{x})}(1.06)^{-i}$$

We calculate this for the possible retirement ages.

x	a_x
60	16.78824
60.5	16.76462
61.5	16.71560
62	16.69016
62.5	16.66409
63.5	16.60999
64.5	16.55319
65	16.52375

We construct the following summary:

t	${}_tp^{(00)}$	1	2	3	Final ave salary	Exp. D. Ben.	a_x	Exp. ret. ben. (w) (with COLA)	Exp. D. ben. (w)	Exp. ret. ben
0	10000.00	118.76	0	0.51	121999.18	9.06	16.52375	1204.06	4.77	
1	9880.73	112.29	0	0.58	126187.11	10.06	16.52375	1223.77	4.85	
2	9767.86	107.16	0	0.65	130538.24	11.00	16.52375	1255.57	4.98	
3	9660.05	101.84	0	0.73	135059.59	12.06	16.52375	1283.05	5.08	
4	9557.49	96.80	0	0.82	139758.48	13.23	16.52375	1311.54	5.20	
5	9459.86	92.02	0	0.93	144642.56	14.64	16.52375	1341.03	5.31	
6	9366.91	87.50	0	1.04	149719.81	15.99	16.52375	1371.77	5.44	
7	9278.37	83.19	0	1.18	154998.60	17.72	16.52375	1403.24	5.56	
8	9193.99	80.11	0	1.32	160487.71	19.36	16.52375	1454.12	5.76	
9	9112.57	75.21	0	1.49	166196.30	21.35	16.52375	1469.31	5.82	
10	9035.87	71.48	0	1.68	172133.95	23.53	16.52375	1503.19	5.96	
11	8962.71	67.92	0	1.89	178310.67	25.87	16.52375	1537.76	6.09	
12	8892.90	64.51	0	2.12	184736.96	28.36	16.52375	1572.71	6.23	
13	8826.26	61.23	0	2.39	191423.80	31.25	16.52375	1607.64	6.37	
14	8762.64	58.07	0	2.69	198382.67	34.39	16.52375	1642.28	6.51	
15	8701.88	55.03	0	3.03	205625.58	37.88	16.52375	1676.64	6.64	
16	8643.83	52.06	0	3.41	213165.06	41.69	16.52375	1709.07	6.77	
17	8588.36	49.18	0	3.84	221014.30	45.92	16.52375	1739.92	6.89	
18	8535.34	46.37	0	4.32	229187.08	50.54	16.52375	1768.23	7.01	
19	8484.64	43.62	0	4.86	237697.81	55.63	16.52375	1793.17	7.11	
10	8436.17	40.90	0	5.47	246561.59	61.27	16.52375	1812.86	7.18	
21	8389.80	38.21	0	6.15	255794.21	67.42	16.52375	1826.42	7.24	
22^-	8345.44		959.64			0.00	16.78824	0.00	0.00	58233.13
22	7385.80	21.70	119.91	5.79	265412.21	62.13	16.76462	1118.77	4.43	7190.48
23	7238.40	18.30	108.44	6.38	275432.90	67.03	16.71560	1017.80	4.03	6347.60
24^-	7105.28		1203.54			0.00	16.69016	0.00	0.00	69591.24
24	5901.74	10.81	384.29	5.86	285874.42	60.28	16.66409	648.71	2.57	21957.98
25	5500.78	9.14	639.20	6.15	296755.74	61.95	16.60999	591.91	2.35	35651.33
26	4846.29	7.73	351.32	6.10	308096.73	60.19	16.55319	540.33	2.14	19126.53
27^-	4481.14		4481.14		318436.83	0.00	16.52375	0.00	0.00	240978.83
Total						959.79		37424.85	148.29	459077.11

The EPV of the accrued benefit is therefore $959.79 + 37424.85 + 459077.11 + 148.29 = \$497610.05$.

5. *An individual aged 39 has 13 years of service, and last year's salary was $119,000. The salary scale is $s_y = 1.04^y$. The accrual rate is 0.01. The interest rate is $i = 0.06$. There is no death benefit, and no exits other than death or retirement at age 65. Mortality follows a Gompertz law with $B = 0.0000023$ and $C = 1.093$. Calculate this year's employer contribution to the plan using*

   (a) The projected unit method.

   The probabilty of surviving to age 65 is $e^{-\frac{0.0000023}{\log(1.093)}(1.093^{65}-1.093^{39})} = 0.9924825698$.

   The EPV of the pension benefit at the time of retirement is

$$\sum_{i=0}^{\infty} e^{-\frac{0.0000023}{\log(1.093)}(1.093^{65+i}-1.093^{65})}(1.06)^{-i} = 16.1365803$$

Using the projected unit method, the final average salary is $119000\left(\frac{1.04^{23}+1.04^{24}+1.04^{25}}{3}\right) = \$317397.21$, so the EPV of the accrued pension benefit is

$$0.9924825698 \times 16.1365803 \times 317397.21 \times 0.13(1.06)^{-26} = 145254.06$$

If the employee survives for another year, the probability of surviving to age 65 is $e^{-\frac{0.0000023}{\log(1.093)}(1.093^{65}-1.093^{40})} = 0.9925591483$.

The EPV (at the current time) of the accrued pension benefit conditional on the employee surviving the current year is

$$0.9925591483 \times 16.1365803 \times 317397.21 \times 0.14(1.06)^{-26} = 156439.54$$

The EPV (at the current time) of the accrued pension benefit next year is

$$156439.54 p_{39}^{00} = 156439.54 e^{-\frac{0.0000023}{\log(1.093)}(1.093^{40}-1.093^{39})} = 156439.54 \times 0.9999228474 = 156427.47$$

This year's employer contribution should be the change in EPV, which is $156427.47 - 145254.06 = \$11,173.39$.

(b) The traditional unit method.

Using the traditional unit method, this year's final average salary is $119000\frac{(1+1.04^{-1}+1.04^{-2})}{3} = 114481.76$.

The EPV of the accrued pension benefit is

$$0.9924825698 \times 16.1365803 \times 114481.76 \times 0.13(1.06)^{-26} = 52391.58$$

Next year, the final average salary is $119000\frac{(1.04+1+1.04^{-1})}{3} = 119061.03$.

The EPV of the accrued pension benefit is

$$0.9925591483 \times 16.1365803 \times 119061.03 \times 0.14(1.06)^{-26} = 58683.10$$

The employer's contribution is therefore $58683.10 - 52391.58 = \$6,291.52$

Bonus Question

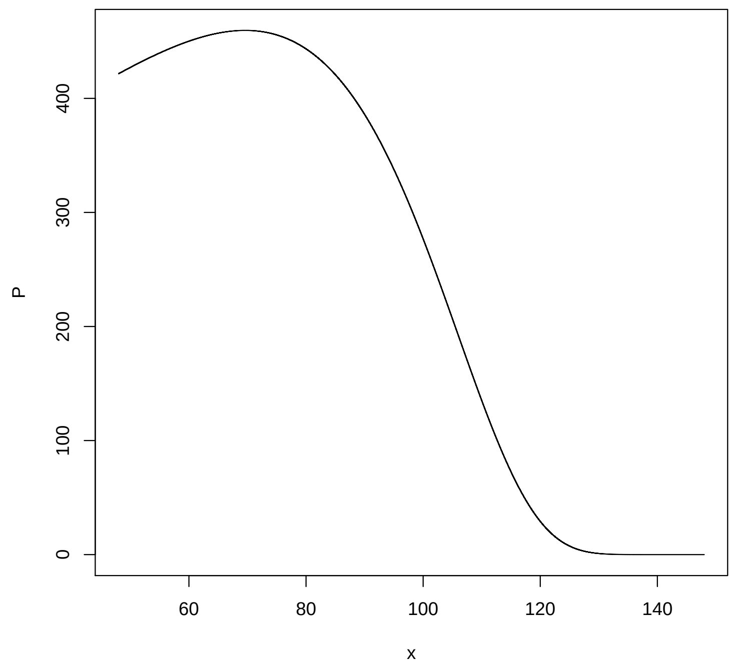
6. *Let the salary scale be $s_y = 1.07^y$, the interest rate be $i = 0.06$, and mortality be $\mu_x = 0.00000204(1.099)^x$. At what retirement age is the EPV of the accrued benefit of a defined benefit plan maximised for an individual aged exactly 48? [Assume the benefits are paid continuously.]*

   Let S be the current final average salary. For retirement age x, the final average salary is $S(1.07)^{x-48}$. The EPV of the pension plan is therefore

$$P = C(1.07)^x \int_x^\infty e^{-\frac{0.00000204}{\log(1.099)}(1.099^t-1.099^{48})}(1.06)^{48-t}\,dt$$

where $C = (1.07)^{-48} Sn\alpha$.

We plot this against x to get the following:



Numerically evaluating this gives the age that maximises the EPV as $x = 69.60945$.