MATH/STAT 4720, Life Contingencies II
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Toby Kenney

In Class Examples

8 Multiple State Models

"Definition"

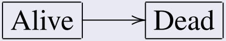
A Multiple State model has several different states into which individuals can be classified. These typically represent different payouts made under the policy.

Examples of Multiple State Models

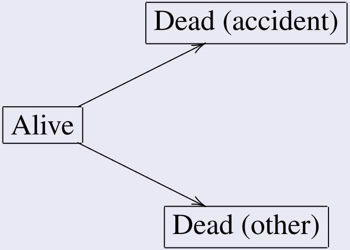
- Alive-Dead.
- Insurance with Increased Benefit for Accidental Death
- Permanent Disability Model.
- Disability Income Insurance Model

8.2 Examples of Multiple State Models

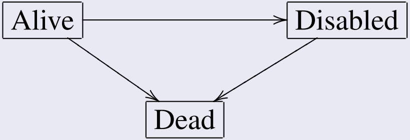
Alive-Dead



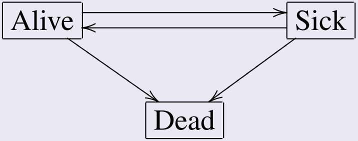
Accidental Death



Permanent Disability



Disability Income



8.4 Assumption and Notation

Assumption [Markov Property]

The probability of any future state depends only on the current state, and not on any information about the process before the present time. Formally:

$$P(Y(x+t) = n | Y(x)) = P(Y(x+t) = n | \{Y(z), z \leqslant x\})$$

Other Assumptions

- The probability of a given transition occuring in a time interval of length t is a differentiable function of t. Effectively, this means that the time at which a transition occurs is a continuous random variable, with no probability mass at any point.
- The probability of two transitions occuring within a time period t tends to zero faster than t.

Notation and Formulae

Notation

${}_tp_x^{ij}$	Probability of going from state i at age x to state j at age $x+t$
${}_tp_x^{\overline{ii}}$	Probability of remaining in state i for a period t for an individual aged x.
μ_x^{ij}	Rate of changing from state i to state j for an individual aged x in state i ($i \neq j$).

Formulae

- ${}_tp_x^{ij} = (Y(x+t) = j | Y(x) = i)$
- ${}_tp_x^{\overline{ii}} = P(Y(x+s) = i \text{ for all } 0 \leqslant s \leqslant t | Y(x) = i)$
- $\mu_x^{ij} = \lim_{t \to 0^+} \frac{{}_tP_x^{ij}}{t}$

8.4 Formulae for Probabilities

- ${}_t p_x^{\overline{ii}} = e^{-\int_x^{x+t} \sum_{j \neq i} \mu_y^{ij} dy}$
- ${}_{t+s} p_x^{ij} = \sum_k {}_t p_x^{ik} \, {}_s p_{x+t}^{kj}$
- $\frac{d}{dt} {}_t p_x^{ij} = \sum_{k \neq j} {}_t p_x^{ik} \mu_x^{kj} - {}_t \, p_x^{ij} \mu_x^{jk}$

Question 1

Under a permanent disability model, with transition intensities

$$\mu_x^{01} = 0.003$$
$$\mu_x^{02} = 0.001$$
$$\mu_x^{12} = 0.002$$

calculate the probability that an individual aged 27 is alive but permanently disabled at age 43.

Question 2

Under a permanent disability model, with transition intensities

$$\mu_x^{01} = 0.003 + 0.000002x$$
$$\mu_x^{02} = 0.001 + 0.000001x$$
$$\mu_x^{12} = 0.002 + 0.000002x$$

calculate the probability that an individual aged 32 is alive but permanently disabled at age 44.

Question 3

Under a disability income model, with transition intensities

$$\mu_x^{01} = 0.0003$$
$$\mu_x^{10} = 0.00003$$
$$\mu_x^{02} = 0.0001$$
$$\mu_x^{12} = 0.0002$$

calculate the probability that an individual aged 27 is alive but disabled at age 43.

Question 4

A disability income model has transition intensities

$\mu_x^{01} = 0.002 \quad \mu_x^{10} = 0.001 \quad \mu_x^{02} = 0.002 \quad \mu_x^{12} = 0.004$

State 0 is healthy, State 1 is sick and State 2 is dead. Three actuaries calculate different values for the transition probabilities and benefit values. Which one has calculated plausible values?

Value	Actuary I	Actuary II	Actuary III
${}_2p_{37}^{(00)}$	0.992036	0.992036	0.992036
${}_2p_{37}^{(01)}$	0.003960	0.003968	0.003964
${}_4p_{37}^{(01)}$	0.007857	0.007857	0.007857
${}_4p_{37}^{(02)}$	0.015857	0.008000	0.008000
${}_4p_{37}^{(12)}$	0.008000	0.015857	0.015857
${}_2p_{39}^{(01)}$	0.003960	0.003968	0.003964
${}_2p_{39}^{(11)}$	0.992054	0.992054	0.990054

Benefit and Annuity functions

$\bar{a}_x^{ij}$	EPV of an annuity paying continuously at a rate of $1 per year, whenever the life is in state j, to a life currently aged x and in state i	$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t} {}_tp_x^{ij} dt$
$\bar{A}_x^{ij}$	EPV of a benefit which pays $1 immediately, whenever the life transitions into state j, to a life currently aged x and in state i	$\bar{A}_x^{ij} = \int_0^\infty \sum_{k \neq j} e^{-\delta t} {}_tp_x^{ik} \mu_{x+y}^{kj} dt$

Question 5

Under a permanent disability model, with transition intensities

$$\mu_x^{01} = 0.0003 + 0.000002x$$
$$\mu_x^{02} = 0.0001 + 0.000001x$$
$$\mu_x^{12} = 0.02$$

The interest rate is $\delta = 0.03$. Calculate the premium for a 5-year policy sold to a life aged 42, with premiums payable continuously while healthy, benefits at a rate of \$90,000 per year are payable while the life is sick, and a death benefit of \$100,000 payable immediately upon death.

Question 6

Under a disability income model, transition intensities are:

$$\mu_x^{01} = 0.0003 + 0.000002x$$
$$\mu_x^{10} = 0.00003 + 0.000001x$$
$$\mu_x^{02} = 0.0001 + 0.000001x^2$$
$$\mu_x^{12} = 0.0002 + 0.000002x$$

The interest rate is $i = 0.06$. Calculate the premium for a 10-year policy sold to a life aged 37, with premiums payable annually in advance while healthy, benefits of $80,000 per year in arrear are payable if the life is sick at the end of a given year, and a death benefit of $200,000 is payable at the end of the year of death.

Answer to Question 6

We calculate the probability that the life is in each state at the end of each year:

t	${}_tp_{37}^{00}$	${}_tp_{37}^{01}$	${}_tp_{37}^{02}$
0	1	0	0
1	0.99812	0.000375	0.001505
2	0.99617	0.000750	0.003083
3	0.99414	0.001127	0.004736
4	0.99203	0.001505	0.006464
5	0.98985	0.001884	0.008271
6	0.98758	0.002263	0.010156
7	0.98523	0.002644	0.012123
8	0.98280	0.003025	0.014171
9	0.98029	0.003407	0.016303
10	0.97769	0.003790	0.018519

Question 7

An insurance company is developing a new model for transition intensities in a disability income model. Under these transition intensities it calculates

$$\bar{A}^{02}_{34} = 0.14 \qquad \bar{A}^{02}_{44} = 0.19 \qquad \bar{A}^{12}_{44} = 0.21$$

$$\bar{a}^{00}_{34} = 22.07 \qquad \bar{a}^{00}_{44} = 19.30 \qquad \bar{a}^{10}_{44} = 0.11$$

$$\bar{a}^{01}_{34} = 0.64 \qquad \bar{a}^{01}_{44} = 0.43 \qquad \bar{a}^{11}_{44} = 17.32$$

$${}_{10}p^{00}_{34} = 0.934 \qquad {}_{10}p^{01}_{34} = 0.022 \qquad \delta = 0.03$$

Calculate the premium for a 10-year policy for a life aged 34, with continuous premiums payable while in the healthy state, which pays a continuous benefit while in the sick state, at a rate of $80,000 per year, and pays a death benefit of $280,000 immediately upon death.

Question 8

A disability income model has the following four states:

State	Meaning	State	Meaning
0	Healthy	2	Accidental Death
1	Sick	3	Other Death

The transition intensities are:

$$\mu_x^{01} = 0.001 \quad \mu_x^{02} = 0.002 \quad \mu_x^{03} = 0.001$$
$$\mu_x^{10} = 0.002 \quad \mu_x^{12} = 0.001 \quad \mu_x^{13} = 0.003$$

t years from the start of the policy, the probability that the life is healthy is $0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t}$; the probability that it is sick is $0.2886752e^{-0.003267949t} - 0.2886752e^{-0.006732051t}$. Calculate the premium for a 5-year policy with premiums payable continuously while the life is in the healthy state, which pays no benefits while the life is in the sick state, but pays a benefit of \$200,000 in the event of accidental death and a benefit of \$100,000 in the event of other death. The interest rate is $\delta = 0.03$.

Thiele's Differential Equation

$$\frac{d}{dt}\,{}_tV^{(i)} = \delta\,{}_tV^{(i)} + P^{(i)} - B^{(i)} - \sum_{j\neq i} \mu_{x+t}^{(ij)}(S^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)})$$

where:

- δ is force of interest.
- ${}_tV^{(i)}$ is the policy value at time t if the life is in state i.
- $P^{(i)}$ is the rate at which premiums are paid while in state i.
- $B^{(i)}$ is the rate at which benefits are paid while in state i.
- $S^{(ij)}$ is the benefit which is paid upon every transition from state i to state j.

Question 9

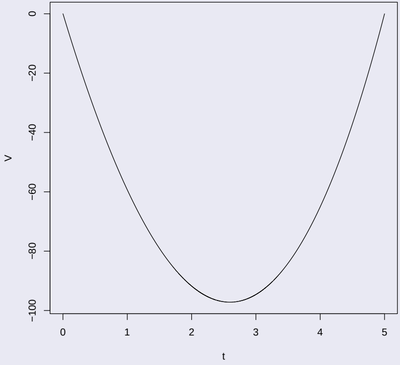
Under a permanent disability model, with transition intensities

$$\mu_x^{01} = 0.0003 + 0.000002x$$
$$\mu_x^{02} = 0.0001 + 0.000001x^2$$
$$\mu_x^{12} = 0.02$$

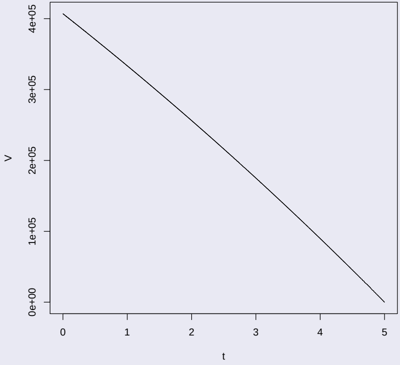
The interest rate is $\delta = 0.03$. Recall (Question 5) that the continuous premium for a 5-year policy sold to a life aged 42 is \$98.54 per year; a benefit at a rate \$90,000 per year is payable while the life is disabled; and a benefit of \$100,000 is payable immediately upon death. Calculate the policy value of this policy in 3 years time, while the life is healthy, and while the life is disabled.

8.7 Policy Values and Thiele's Differential Equation

Answer to Question 9



(a) Healthy



(b) Disabled

Question 10

In a certain life insurance policy, mortality is modelled as $\mu_x = 0.0003 + 0.00002x$, while policies lapse at a rate $\lambda_x = 0.002 - 0.00001x$. Force of interest is $\delta = 0.04$. Calculate the continuous premium for a 10-year policy with death benefits \$300,000, payable immediately on death sold to a life aged 36.

(a) If the insurer makes no payments to policies which lapse.

(b) If policies can be surrenderred for half the policy value. [Policy value is calculated under the assumption that the policy does not lapse.]

Question 11

A certain life insurance policy, pays double benefits for accidental death (state 1). Mortality is modelled as

$$\begin{aligned} \mu_x^{01} &= 0.0003 \\ \mu_x^{02} &= 0.00002x \\ \mu_x^{03} &= 0.002 - 0.00001x \end{aligned}$$

Where state 1 represents accidental death, state 2 represents other deaths, and state 3 represents lapse. [The insurer makes no payments to policies which lapse.] Calculate the continuous premium for a 10-year policy with death benefits $400,000 for accidental death, and $200,000 for other deaths, payable immediately on death sold to a life aged 29, if force of interest is $\delta = 0.05$.

Question 12

The following is a multiple decrement table, giving probabilities of surrender, accidental death, and other death.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
40	10000.00	59.00	0.30	1.62
41	9939.08	58.65	0.29	1.70
42	9878.44	58.31	0.28	1.78
43	9818.06	57.96	0.27	1.89
44	9757.95	57.62	0.27	1.98
45	9698.08	57.28	0.26	2.10
46	9638.44	56.94	0.25	2.23
47	9579.02	56.61	0.24	2.36
48	9519.81	56.27	0.24	2.51
49	9460.78	55.94	0.23	2.68

Calculate the probability that a life who purchases a policy at age 42 surrenders it between ages 46 and 48.

Question 13

The following is a multiple decrement table, giving probabilities of surrender, accidental death, and other death.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
40	10000.00	59.00	0.30	1.62
41	9939.08	58.65	0.29	1.70
42	9878.44	58.31	0.28	1.78
43	9818.06	57.96	0.27	1.89
44	9757.95	57.62	0.27	1.98
45	9698.08	57.28	0.26	2.10

An annual 5-year term annual insurance policy pays benefits of \$200,000 in the case of accidental death, \$100,000 in the case of other death, and has no surrender value. Calculate the net premiums for this policy sold to a life aged 40 at interest rate $i = 0.03$.

Question 14

Recall the multiple decrement table from Question 12, giving probabilities of surrender, accidental death, and other death.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
40	10000.00	59.00	0.30	1.62
41	9939.08	58.65	0.29	1.70
42	9878.44	58.31	0.28	1.78
43	9818.06	57.96	0.27	1.89
44	9757.95	57.62	0.27	1.98

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
45	9698.08	57.28	0.26	2.10
46	9638.44	56.94	0.25	2.23
47	9579.02	56.61	0.24	2.36
48	9519.81	56.27	0.24	2.51
49	9460.78	55.94	0.23	2.68

Calculate the probability that a life who purchases a policy at age 42 and 4 months dies in an accident between ages 46 and 3 months and 47 and 5 months using:

(a) UDD

(b) Constant transition intensities.

Question 15

You want to update the multiple decrement table on the left below with the updated mortalities from the table on the right.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
40	10000.00	59.00	1.92
41	9939.08	58.65	1.99
42	9878.44	58.31	2.06
43	9818.06	57.96	2.16
44	9757.95	57.62	2.25
45	9698.08	57.28	2.36
46	9638.44	56.94	2.48
47	9579.02	56.61	2.60

x	l_x	d_x
40	10000.00	1.10
41	9998.90	1.18
42	9997.72	1.26
43	9996.46	1.35
44	9995.11	1.45
45	9993.66	1.56
46	9992.10	1.67
47	9990.43	1.80

Construct the new multiple decrement table using:

(a) UDD in the Multiple Decrement table.
(b) Constant transition probabilities.
(c) UDD in the independent models

Answer to Question 15

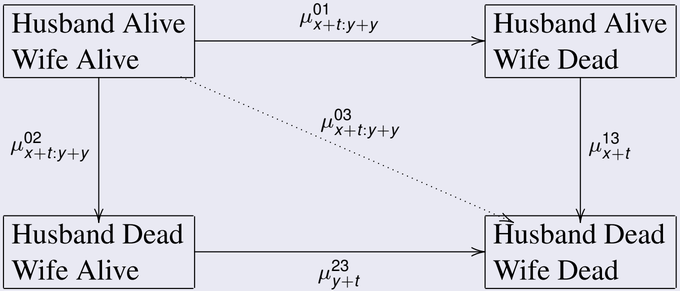
(a) and (b)

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
40	10000.00	59.00	1.10
41	9939.90	58.66	1.17
42	9880.07	58.32	1.24
43	9820.51	57.98	1.32
44	9761.21	57.64	1.41
45	9702.16	57.31	1.51
46	9643.34	56.97	1.61
47	9584.76	56.65	1.72

(c)

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
40	10000.00	59.00	1.10
41	9939.90	58.66	1.17
42	9880.07	58.32	1.24
43	9820.51	57.98	1.32
44	9761.21	57.64	1.41
45	9702.16	57.31	1.51
46	9643.34	56.97	1.61
47	9584.76	56.65	1.72

Model


Husband Alive
Wife Alive
$\mu^{01}_{x+t:y+y}$
Husband Alive
Wife Dead
$\mu^{02}_{x+t:y+y}$
$\mu^{03}_{x+t:y+y}$
μ^{13}_{x+t}
Husband Dead
Wife Alive
μ^{23}_{y+t}
Husband Dead
Wife Dead

9.2 Joint Life and Last Survivor Benefits

Joint Policies

Joint life annuity	a_{xy}	pays regular payments while both lives are alive.
Joint life insurance	A_{xy}	pays a death benefit upon the death of either life.
Last survivor annuity	$a_{\overline{xy}}$	pays regular payments while either life is still alive.
Last survivor insurance	$A_{\overline{xy}}$	pays a death benefit upon the death of both lifes.
Reversionary annuity	$a_{x\|y}$	pays regular payments while husband is dead and wife is alive.
Contingent insurance	A^{1}_{xy}	pays a death benefit upon the death of husband provided wife is alive.

Question 16

A couple want to receive a pension of \$200,000 per year while both are alive. If the husband is alive, but the wife is not, he wants to receive \$60,000 per year. If the wife is alive, but the husband is not, she wants to receive \$220,000 per year. When they both die, they want to leave an inheritance of \$700,000 to their children. Construct a collection of insurance and annuity policies that will achieve these objectives.

Question 17

What are the advantages and disadvantages of a reversionary annuity over a standard life insurance policy, whose benefit could be used to purchase an annuity at the time the life dies.

Formulae

$$a_{\overline{xy}} = a_x + a_y - a_{xy}$$

$$a_{x|y} = a_y - a_{xy}$$

$$A_{\overline{xy}} = A_x + A_y - A_{xy}$$

$$A_{x|y} + A_{y|x} = A_{xy}$$

$$\bar{a}_{xy} = \frac{1 - \bar{A}_{xy}}{\delta}$$

Assumptions

- While both husband and wife are alive, the probability of dying depends on both ages.
- Once one life has died, the probability of the other life dying depends on the age of that life and the fact that the other life has died, but not the time the other life died, or the age before they died.

9.3 Joint Life Notation

Standard Notation for Joint Life Probabilities

Notation	Meaning	Multi-state
${}_tp_{xy}$	Probability both still alive at time t	${}_tp_{xy}^{00}$
${}_tq_{xy}$	Probability not both still alive at time t	$1 -_t p_{xy}^{00}$
${}_tq_{xy}^1$	Probability husband dies first before time t	
${}_tq_{xy}^2$	Probability husband dies second before time t	
${}_tp_{\overline{xy}}$	Probability at least one still alive at time t	$1 -_t p_{xy}^{03}$
${}_tq_{\overline{xy}}$	Probability both dead at time t	${}_tp_{xy}^{03}$

$$ {}_tq_{xy}^1 =_t p_{xy}^{02} + \int_0^t {}_sp_{xy}^{00}\mu_{x+s:y+s}^{02}\,{}_{t-s}p_y^{23}\,ds $$

$$ {}_tq_{xy}^2 = \int_0^t {}_sp_{xy}^{00}\mu_{x+s:y+s}^{01}\,{}_{t-s}p_x^{13}\,ds $$

Formulae

$$\bar{a}_{xy} = \int_0^\infty e^{-\delta t} {}_tp_{xy}^{00} dt$$

$$\bar{A}_{xy} = \int_0^\infty e^{-\delta t} {}_tp_{xy}^{00} (\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02}) dt$$

$$\bar{a}_{\overline{xy}} = \int_0^\infty e^{-\delta t} ({}_tp_{xy}^{00} +_t p_{xy}^{01} +_t p_{xy}^{02}) dt$$

$$\bar{A}_{\overline{xy}} = \int_0^\infty e^{-\delta t} ({}_tp_{xy}^{00} \mu_{x+t:y+t}^{03} +_t p_{xy}^{01} \mu_{x+t}^{13} +_t p_{xy}^{02} \mu_{y+t}^{23}) dt$$

$$\bar{a}_{x|y} = \int_0^\infty e^{-\delta t} {}_tp_{xy}^{02} dt$$

$$\bar{A}_{xy}^{1} = \int_0^\infty e^{-\delta t} {}_tp_{xy}^{00} \mu_{x+t:y+t}^{02} dt$$

Question 18

A husband is 63. His wife is 62. Their mortalities both follow the lifetable below, and are assumed to be independent. They purchase a 10-year last survivor insurance policy with a death benefit of $2000,000. Annual Premiums are payable while both are alive. Calculate the net premiums using the equivalence principle and interest rate $i = 0.07$.

x	l_x	d_x
62	10000.00	1.70
63	9998.30	1.83
64	9996.47	1.98
65	9994.49	2.14
66	9992.35	2.31
67	9990.03	2.50

x	l_x	d_x
68	9987.53	2.70
69	9984.83	2.92
70	9981.91	3.16
71	9978.76	3.41
72	9975.34	3.69
73	9971.66	3.99

Question 19

A husband is 53. His wife is 64. Their independent mortalities both follow the lifetables below. They purchase a 7-year reversionary annuity. Annual Premiums are payable while both are alive. If the husband dies first, the policy will provide a life annuity to the wife with annual payments of $30,000. The lifetables are given below. Calculate the net premiums for this policy using the equivalence principle and an interest rate $i = 0.05$. For the wife, we have $\ddot{a}_{71} = 13.89755$.

x	l_x	d_x
53	10000.00	1.35
54	9998.65	1.48
55	9997.17	1.61
56	9995.55	1.76
57	9993.80	1.92
58	9991.87	2.10
59	9989.78	2.29

x	l_x	d_x
64	10000.00	2.64
65	9997.36	2.88
66	9994.48	3.14
67	9991.34	3.42
68	9987.91	3.73
69	9984.19	4.06
70	9980.12	4.43

Question 20

A husband is 72. His wife is 48. They purchase a last survivor annuity which pays $45,000 a year. The life-tables are below. Calculate the net premium for this insurance at $i = 0.06$. For the wife, $\ddot{a}_{68} = 16.1807$.

x	l_x	d_x
72	10000.00	576.84
73	9423.16	631.08
74	8792.08	683.61
75	8108.48	731.95
76	7376.52	773.08
77	6603.44	803.48
78	5799.96	819.33
79	4980.63	816.87
80	4163.76	792.84
81	3370.92	745.21
82	2625.71	673.92
83	1951.79	581.60
84	1370.19	474.03
85	896.16	359.95

x	l_x	d_x
86	536.21	250.05
87	286.16	154.93
88	131.23	82.49
89	48.74	35.57
90	13.17	11.16
91	2.01	1.98
92	0.03	0.03

x	l_x	d_x
48	10000.00	1.03
49	9998.97	1.10
50	9997.87	1.18
51	9996.69	1.26
52	9995.44	1.35
53	9994.09	1.44

x	l_x	d_x
54	9992.65	1.55
55	9991.10	1.66
56	9989.43	1.79
57	9987.65	1.92
58	9985.72	2.07
59	9983.65	2.23
60	9981.42	2.40
61	9979.02	2.59
62	9976.43	2.80
63	9973.63	3.02
64	9970.61	3.26
65	9967.35	3.52
66	9963.83	3.81
67	9960.02	4.12
68	9955.90	4.46

Question 21

A husband is 45. His wife is 76. Their lifetables are below. They purchase a 7-year joint life insurance policy with a death benefit of $850,000. If the interest rate is $i = 0.04$, calculate the monthly net premiums for this policy using the equivalence principle and the UDD assumption.

x	l_x	d_x
45	10000.00	1.80
46	9998.20	1.93
47	9996.26	2.08
48	9994.18	2.23
49	9991.95	2.40
50	9989.55	2.58
51	9986.97	2.78
52	9984.19	3.00

x	l_x	d_x
76	10000.00	16.51
77	9983.49	17.85
78	9965.64	19.29
79	9946.34	20.85
80	9925.49	22.54
81	9902.95	24.35
82	9878.60	26.31
83	9852.28	28.43

9.6 A Model with Dependent Future Lifetimes

Why are joint lives not independent?

- Broken heart syndrome.
- Common accident or illness.
- Similar lifestyles.

Question 22

A husband is 84. His wife is 39. Their mortalities while both are alive and the wife's mortality after the husband has died are shown below. What is the probability that the wife dies within 10 years?

x	l_x	d_x	x	l_x	d_x	x	l_x	d_x
84	10000.00	45.99	39	10000.00	1.00	39	10000.00	2.53
85	9954.01	49.84	40	9999.00	1.06	40	9997.47	2.75
86	9904.17	53.98	41	9997.94	1.13	41	9994.72	2.99
87	9850.19	58.44	42	9996.81	1.20	42	9991.74	3.25
88	9791.76	63.24	43	9995.60	1.28	43	9988.49	3.54
89	9728.52	68.40	44	9994.32	1.37	44	9984.95	3.85
90	9660.11	73.94	45	9992.94	1.47	45	9981.10	4.19
91	9586.17	79.89	46	9991.48	1.57	46	9976.91	4.57
92	9506.28	86.25	47	9989.91	1.68	47	9972.34	4.98
93	9420.03	93.05	48	9988.23	1.80	48	9967.36	5.43
94	9326.98	100.31	49	9986.44	1.93	49	9961.92	5.93

(a) Assuming changes to the wife's mortality apply at the end of the year of the husband's death.

(b) Using the UDD assumption.

Question 23

For the couple in Question 22 (lifetables recalled below). What is the premium for a 10-year annual life insurance policy for the wife with benefit $200,000 at interest rate $i = 0.04$. [Use the UDD assumption for changes to the wife's mortality at time of the Husband's death.]

x	l_x	d_x
84	10000.00	45.99
85	9954.01	49.84
86	9904.17	53.98
87	9850.19	58.44
88	9791.76	63.24
89	9728.52	68.40
90	9660.11	73.94
91	9586.17	79.89
92	9506.28	86.25
93	9420.03	93.05
94	9326.98	100.31

x	l_x	d_x
39	10000.00	1.00
40	9999.00	1.06
41	9997.94	1.13
42	9996.81	1.20
43	9995.60	1.28
44	9994.32	1.37
45	9992.94	1.47
46	9991.48	1.57
47	9989.91	1.68
48	9988.23	1.80
49	9986.44	1.93

x	l_x	d_x
39	10000.00	2.53
40	9997.47	2.75
41	9994.72	2.99
42	9991.74	3.25
43	9988.49	3.54
44	9984.95	3.85
45	9981.10	4.19
46	9976.91	4.57
47	9972.34	4.98
48	9967.36	5.43
49	9961.92	5.93

Question 24

A husband aged 25 and a wife aged 56 have the following transition intensities:

$$\mu_{xy}^{01} = 0.000001y^2 + 0.000000001x$$
$$\mu_{xy}^{02} = 0.000002x^2 + 0.000000002y$$
$$\mu_{xy}^{03} = 0.000042$$
$$\mu_{x}^{13} = 0.000003x^2$$
$$\mu_{y}^{23} = 0.000002y^2$$

Calculate the probability that in ten years time the husband is dead, and the wife is still alive.

Question 25

A husband aged 25 and a wife aged 56 have the following transition intensities:

$$\mu_{xy}^{01} = 0.000001y^2 + 0.000000001x$$
$$\mu_{xy}^{02} = 0.000002x^2 + 0.000000002y$$
$$\mu_{xy}^{03} = 0.000042$$
$$\mu_{x}^{13} = 0.000003x^2$$
$$\mu_{y}^{23} = 0.005$$

They wish to purchase a reversionary annuity, which will provide a continuous life annuity to the wife at a rate of $25,000 per year after the husband's death. The premiums are payable continuously while both are alive. The interest rate is $\delta = 0.04$. Calculate the rate of premiums.

Question 26

A husband aged 75 and a wife aged 29 have the following transition intensities:

$$\mu_{xy}^{01} = 0.001y + 0.000001x$$
$$\mu_{xy}^{02} = 0.002x + 0.000002y$$
$$\mu_{xy}^{03} = 0.012$$
$$\mu_{x}^{13} = 0.003x$$
$$\mu_{y}^{23} = 0.002y$$

They wish to purchase an annual whole-life last survivor insurance policy with benefit $300,000. The interest rate is $i = 0.06$.
(a) Calculate the annual premiums. (Premiums are payable while either life is still alive).
(b) Calculate the policy value after 10 years if the husband is dead, but the wife is alive.

10.2 Introduction to Pensions

Reasons for Employers Offering Pensions

- Competition for new employees
- Facilitate retirement of older employees.
- Provide an incentive for employees to remain with the organisation.
- Pressure from trade unions.
- Tax efficiency
- Social Responsibility

Types of Pension Plan

Defined Contribution

- Employer contributions specified.
- Employee contributions may be permitted, and may influence employer contributions according to some formula (e.g. matching contributions)
- Contributions held in an account.
- Employee receives account upon retirement.
- Retirement benefits depend on state of the account when employees retire.
- Contributions may be designed to achieve a target level of retirement benefits. Actual benefits may be different from target benefits.

Types of Pension Plan

Defined Benefit

- Retirement benefit specified according to a formula usually based on:
  - Final or average salary
  - Years of service
- Contributions may need to be adjusted according to performance of investment and mortality experience.
- Funding is monitored on a regular basis to assess whether contributions need to be changed.

10.3 The Salary Scale Function

Estimating Future Salary

- Salary scale is given by a function s_y.
- If salary at age x is P, salary at age $y > x$ for an employee who remains employed at the company between ages x and y is $\frac{s_y}{s_x}P$.
- In practice, salary is more uncertain, but this model is widely used.
- It is important to make a distinction between salary in the year between ages x and $x+1$ and salary rate at age x. The latter is usually approximated as the salary received between age $x-0.5$ and age $x+0.5$.

Question 27

An individual aged 42 has a current salary of $60,000 (i.e. salary in the year from age 42 to 43 is $60,000). Estimate her final average salary (average over last 3 years working) assuming she retires at age 65 if:

(a) The salary scale is given by $s_y = 1.03^y$.

(b) The salary scale at integer ages is as shown in the table below:

x	s_x	x	s_x	x	s_x	x	s_x
42	1.000	49	1.391	56	1.827	63	2.335
43	1.036	50	1.424	57	1.904	64	2.400
44	1.092	51	1.470	58	1.982		
45	1.164	52	1.515	59	2.056		
46	1.228	53	1.583	60	2.120		
47	1.290	54	1.679	61	2.187		
48	1.334	55	1.748	62	2.261		

(c) What if the individual is currently aged 42 and 4 months?

Question 28

An employer sets up a DC pension plan for its employees. The target replacement ratio is 60% of final average salary for an employee who enters the plan at age exactly 30. Under the following assumptions:

- At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity valued at 50% of the life annuity.
- At age 65, the employee is married to someone aged 62.
- The salary scale is $s_y = 1.03^y$.
- Mortalities are independent and given by $\mu_x = 0.000002(1.093)^x$.
- A fixed percentage of salary is payable monthly in arrear.
- Contributions earn an annual rate of return of 6%.
- The value of a life annuity is based on a rate of interest of 4%.

Calculate the percentage of salary payable monthly.

Question 29

Recall from Question 28, that the rate of contribution was 20.74%. Calculate the actual replacement ratio achieved if the following changes are made to the assumptions:

(a) At age 65, the employee is not married.
(b) At age 65, the employee's spouse is aged 73.
(c) The rate of return on contributions is 7%.
(d) Salary increases continuously at an annual rate of 5%.
(e) At age 65, the employee purchases a whole life annuity, plus a reversionary annuity for only 30% of the value.
(f) The life annuities are valued using an interest rate of 3%.
(g) The employee is in poor health at retirement, and has mortality given by $\mu_x = 0.000002(1.143)^x$. [The employee's spouse still has mortality given by $\mu_x = 0.000002(1.093)^x$.]

Reasons for Early Exit

- Withdrawl — Leaving to take another job (or for other reasons).
- Early retirement.
- Disability retirement.
- Death.

Question 30

For a multiple decrement model with the following states and transition intensities:

0 — Employed
1 — Withdrawn
2 — Disability retirement
3 — Age retirement
4 — Death

$$\mu_x^{(01)} = e^{-0.07x}$$
$$\mu_x^{(02)} = 0.0004$$
$$\mu_x^{(03)} = 0.08 \text{ for } 60 < x < 65$$
$$\mu_x^{(04)} = 0.000002 \times 1.102^x$$

In addition, 25% of employees who reach age 60 retire then, 30% of employees still employed at age 62 retire then, and all employees still working at age 65 retire then.

(a) Construct a service table for ages from 30 to 65.

(b) What is the probability that an employee currently aged exactly 37 retires while aged 63.

Answer to Question 30

t	$_tp^{(00)}$	1	2	3	4
0	10000.00	1182.45	4.00	0	0.39
1	8813.17	971.66	3.53	0	0.38
2	7837.61	805.68	3.14	0	0.37
3	7028.43	673.65	2.81	0	0.36
4	6351.59	567.63	2.54	0	0.36
5	5781.07	481.71	2.31	0	0.36
6	5296.68	411.51	2.12	0	0.37
7	4882.68	353.70	1.95	0	0.37
8	4526.66	305.74	1.81	0	0.38
9	4218.72	265.68	1.69	0	0.39
10	3950.97	231.99	1.58	0	0.40
11	3716.99	203.50	1.49	0	0.42
12	3511.58	179.26	1.40	0	0.44
13	3330.48	158.52	1.33	0	0.46
14	3170.18	140.69	1.27	0	0.48
15	3027.74	125.28	1.21	0	0.50
16	2900.75	111.91	1.16	0	0.53
17	2787.14	100.26	1.11	0	0.56
18	2685.21	90.06	1.07	0	0.60

t	$_tp^{(00)}$	1	2	3	4
19	2593.47	81.11	1.04	0	0.64
10	2510.69	73.21	1.00	0	0.68
21	2435.80	66.22	0.97	0	0.72
22	2367.88	60.02	0.95	0	0.78
23	2306.13	54.51	0.92	0	0.83
24	2249.87	49.58	0.90	0	0.90
25	2198.49	45.17	0.88	0	0.96
26	2151.48	41.22	0.86	0	1.04
27	2108.36	37.66	0.84	0	1.12
28	2068.73	34.46	0.83	0	1.21
29	2032.23	31.56	0.81	0	1.31
30^-	1998.54			499.64	
30	1498.90	21.70	0.60	119.91	1.07
31	1355.62	18.30	0.55	108.44	1.06
32^-	1227.26			368.18	
32	859.02	10.81	0.34	68.73	0.74
33	778.45	9.14	0.31	62.28	0.74
34	705.99	7.73	0.28	56.48	0.74
35^-	640.76		640.76		

Annual Pension Benefit

$$nS_{\text{Fin}}\alpha$$

- n is the number of years of service. (Possibly capped by some upper bound).
- S_{Fin} is the final average salary.
- α is the accrual rate (usually between 0.01 and 0.02).

For an individual aged y who joined the pension at age x, the estimated benefits are often given as

$$(R-x)\hat{S}_{\text{Fin}}\alpha = (y-x)\hat{S}_{\text{Fin}}\alpha + (R-y)\hat{S}_{\text{Fin}}\alpha$$

where R is the normal retirement age for the individual. The first term $(y-x)\hat{S}_{\text{Fin}}\alpha$ is called the accrued benefit. Only accrued benefits are considered liabilities for valuation purposes.

Projected vs. Current Unit Method

- Projected Unit Method uses estimated future salary at retirement.
- Traditional or Current Unit Method uses current final average salary.

Question 31

The salary scale is $s_y = 1.04^y$. A defined benefit pension plan has $\alpha = 0.01$ and S_{Fin} is the average of the last 3 years' salary. A member's mortality follows a Gompertz model with $B = 0.0000023$, $C = 1.12$. The member is currently aged 46, has 13 years of service and the member's annual salary for the coming year is \$76,000. The interest rate is $i = 0.05$. The pension benefit is paid monthly in advance. Calculate the EPV of the accrued benefit under the assumption that:

(a) The individual retires at age 65.

(b) The individual retires at age 60.

(c) The individual's retirement happens between ages 60 and 65. The probability of retirement at 60 is 0.3. Between ages 60 and 65, $\mu_x^{(03)} = 0.15$, and there are no other decrements between these ages. [Calculate the conditional EPV conditioning on the member exiting through retirement. You may use the approximation that retirements not at an exact age happen in the middle of the year of retirement.]

Question 32

An employee aged 43 has been working for a company for 15 years. The salary scale is $s_y = 1.05^y$. The employee's salary last year was $75,000. If the employee withdraws from the pension plan, he receives a deferred pension based on accrual rate 2%, with COLA of 2% per year. He receives the pension starting from age 65 with payments monthly in advance. The individual's mortality is given by $\mu_x^{(04)} = 0.000002 \times 1.102^x$. The interest rate is $i = 0.04$.

(a) Calculate the EPV of the pension benefits if he withdraws now.

(b) Calculate the EPV of the accrued withdrawl benefits if the rate of withdrawl is $\mu_x^{(01)} = e^{-0.07x}$ (conditional on the employee withdrawing before age 60).

Question 33

Let the salary scale be $s_y = 1.04^y$. A pension plan has benefit defined by $\alpha = 0.015$ and S_{Fin} is the average of the last 3 years' salary. Suppose a member's mortality follows a Gompertz model with $B = 0.0000023$, $C = 1.12$. The member is currently aged 46 and has 13 years of service, and a current annual salary of \$45,000. The rate of withdrawl from the pension plan is $\mu_x^{(01)} = e^{-0.07x}$. The individual will retire at age 60 with probability 0.3; will retire at rate $\mu_x^{(03)} = 0.06$ between ages 60 and 65; and will retire at age 65 if still employed at that age. The interest rate is $i = 0.06$ while the employee is employed. Once the employee exits the plan, the benefits are calculated at an interest rate $i = 0.05$. The pension benefit is paid monthly in advance. Upon withdrawl, the employee receives a deferred pension with COLA 2%. There is no death benefit. Calculate the EPV of the accrued benefit of the employee.

Question 34

A pension plan offers a benefit of 4% of career average earnings per year of service. The benefit is payable monthly in advance. Mortality follows a Gompertz model with $B = 0.0000023$, $C = 1.12$. The salary scale is $s_y = 1.04^y$. One plan member aged 44 joined the plan 6 years ago with a starting salary of $180,000. Withdrawls receive a deferred pension benefit. The rate of withdrawl from the pension plan is $\mu_x^{(01)} = e^{-0.07x}$. The individual will retire at age 60 with probability 0.3; will retire at rate $\mu_x^{(03)} = 0.06$ between ages 60 and 65; and will retire at age 65 if still employed at that age. The interest rate is $i = 0.06$ while the employee is employed. Once the employee exits the plan, the benefits are calculated at an interest rate $i = 0.05$. There is no death benefit. Calculate the EPV of the accrued benefit.

Funding DB Pension Plans

- Employee pays fixed contribution (as percentage of salary).
- Employer pays the remaining costs of benefits.
- Employer contributions not usually specified in contract. Employer has an incentive to keep its contributions smooth and predictable.
- Employer will usually establish a reserve level equal to the EPV of accrued liabilities, called Actuarial Liability.

Normal contribution C_t at start of year satisfies

$$ {}_tV + C_t = \text{EPV of benefits for exits during the year} + (1+i)^{-1}\,{}_1p_x^{00}\,{}_{t+1}V $$

Question 35

An individual aged 45 has 26 years of service, and a last year's salary of \$47,000. The salary scale is $s_y = 1.05^y$, and the accrual rate is 0.02. The interest rate is $i = 0.04$. There is no death benefit. There are no exits other than death or retirement at age 65. Mortality follows a Gompertz model with $B = 0.0000076$, $C = 1.087$. Calculate this year's employer contribution to the plan using:

(a) The Projected Unit Method.

(b) The Traditional Unit Method.

Question 36

Annual Pension benefits are 1% of final average salary over 3 years per year of service. The salary scale is $s_y = 1.06^y$. Mortality follows a Gompertz model with $B = 0.00000187$, $C = 1.130$. The rate of withdrawl is $\mu_x^{01} = 0.2e^{-0.04x}$. Withdrawl benefits take the form of a deferred pension with COLA 2%, beginning at age 65. The benefit for death while in service is 3 times the last year's annual salary. Pension benefits are guaranteed for 5 years. Interest rates are 5%. Members alive at age 60 retire then with probability 0.08. Members aged between 60 and 65 retire at a rate $\mu_x^{03} = 0.1$. Members who are still employed at age 65 all retire then. If a member aged 46 has 12 years of service and last year's salary $87,000, and makes an annual contribution of 4% of annual salary, calculate the employer's annual contribution to the pension plan on behalf of this member.

11.2 The Yield Curve

The Yield Curve

Interest rates usually depend on the term of the investment (interest rates can be different depending on how long until maturity). The yield curve summarises this difference. Based on the no-arbitrage principle, it allows us to calculate implied forward rates. Usually these rates can be arranged in advance. That is, an agreement to invest or borrow money at a specific future time at an agreed rate can be made.

Notation

$v(t)$	Present value of t-year zero-coupon bond with face value 1
y_t	Spot rate (yield rate of t-year zero-coupon bond)
Term structure	yield rate as a function of time to maturity
$f(t, t+k)$	Forward rate (annual effective) from time t to $t+k$.

Future cash-flows are valued by applying the appropriate discount to each payment.

Question 37

The yield rate on 3-year zero-coupon bonds is 4.3%. The yield rate on 6-year zero-coupon bonds is 4.8%. What is the forward rate for a 3-year loan starting in 3 years' time?

EPV of Benefits under Non-flat Term Structures

$$\ddot{a}(x)_y = \sum_{k=0}^{\infty} {}_x p_k v(k)$$

$$A(x)_y = \sum_{k=0}^{\infty} {}_x p_k q_k v(k+1)$$

Question 38

A life aged 58 follows the lifetable below. Yield rates are also given below. Calculate the net annual premium for a 5-year term insurance policy with death benefit $300,000 sold to this life.

x	l_x	d_x
58	10000.00	3.38
59	9996.62	3.68
60	9992.94	4.03
61	9988.91	4.39
62	9984.52	4.81
63	9979.71	5.25

term(years)	yield rate
1	0.034
2	0.036
3	0.039
4	0.041
5	0.042

Answer to Question 38

t	$v(t)$	${}_tp_{58}$	${}_{t-1}p_{58}q_{57+t}$	$v(t)\,{}_tp_{58}$	$v(t)\,{}_{t-1}p_{58}q_{57+t}$
0	1	1		1	
1	0.96712	0.9997	0.000338	0.96679	0.00032689
2	0.93171	0.9993	0.000368	0.93105	0.00034287
3	0.89157	0.9989	0.000403	0.89058	0.00035930
4	0.85152	0.9985	0.000439	0.85021	0.00037382
5	0.81407	0.9980	0.000481		0.00039157
Total				4.638626	0.001794442

Question 39

Suppose the company sells 1,000,000 policies identical to the policy in Question 38. Mortality experience perfectly matches the expected mortality, and the company arranges forward rate agreements, so that future interest rates perfectly match the current forward rates, calculate the cash-flows of these policies over time.

Answer to Question 39

Dollar amounts in million dollars:

Year	Premiums	Forward Rate $f(t, t+1)$	Expected claims	Cumulative Net Cash Flow
1	116.05	0.034	101.4	18.60
2	116.02	0.038	110.4	29.33
3	115.97	0.045	120.9	30.95
4	115.93	0.047	131.7	22.08
5	115.87	0.046	144.3	0.00

11.4 Diversifiable and Non-diversifiable Risk

Definition

A risk X_i is diversifiable if

$$\lim_{N\to\infty} \frac{\sqrt{\text{Var}\left(\sum_{i=1}^{N} X_i\right)}}{N} = 0$$

A risk is non-diversifiable if this condition does not hold.

Diversifiable Risks

- Typically independent of one another.
- Can be effectively eliminated by taking a large enough portfolio.

Non-diversifiable Risks

- Cannot be eliminated by taking a larger portfolio.
- Generally represent large-scale economic conditions.

11.4 Diversifiable and Non-diversifiable Risk

When Can Mortality be Treated as Diversifiable?

- Lives are approximately independent.
- Policies are for similar benefits.
- The mortality models used are correct. (Different lives can use different mortality models).

When are Mortality Risks not Fully Diversified?

- For very old ages, the number of policies sold is usually small.
- For policies with a very large benefit, the risks can unduly influence total risk.
- Errors in the mortality model can introduce systematic bias.
- Events like natural disasters, wars, or epidemics can cause abnormal mortality.
- Likewise, health advances can also cause abnormal mortality.

Why do Insurers not use Forward Rates to Remove Interest Rate Risk?

- Fixed rate investments with such long terms may not be available.
- They may be able to obtain better rates on average by taking on more risk.
- For a large insurance company, the amount of risk they need to cover could influence prices.

Question 40

Consider a 10-year term insurance policy sold to a life aged 24 for whom the lifetable below is appropriate, with a death benefit of $3,200,000. Using an interest rate of $i = 0.05$, they calculate a net annual premium of $91.95. Calculate the expected profit or loss on the policy if the interest rate changes after 1 year to:

(a) $i = 0.04$

(b) $i = 0.06$.

[The insurance company invests premiums at the current interest rate for a one-year period each year.]

x	l_x	d_x
24	10000.00	0.23
25	9999.77	0.25
26	9999.52	0.26
27	9999.26	0.28
28	9998.98	0.29

x	l_x	d_x
29	9998.68	0.31
30	9998.37	0.33
31	9998.04	0.35
32	9997.69	0.38
33	9997.31	0.40

Comments on Question 40

- As expected, an increase in interest rates causes a profit, while a decrease causes a loss.
- Selling more policies would not resolve this risk, because each policy has the same interest rate, so each policy would be expected to make a profit or loss.
- The decrease in interest rates causes smaller losses than the profits caused by an increase in interest rates, so if there is some probability of interest rates decreasing, and the same probability of decreasing, taking the average interest rate can result in an expected profit. It can also result in an expected loss.

Question 41

For the policy in Question 40, imagine the insurance company sells N identical policies with premium \$91.95. Suppose that the interest rate in 1 year's time is 0.05 with probability 0.6; 0.04 with probability 0.2; and 0.06 with probability 0.2. Calculate the variance of the present value of the aggregate loss on these polices.

x	l_x	d_x
24	10000.00	0.23
25	9999.77	0.25
26	9999.52	0.26
27	9999.26	0.28
28	9998.98	0.29

x	l_x	d_x
29	9998.68	0.31
30	9998.37	0.33
31	9998.04	0.35
32	9997.69	0.38
33	9997.31	0.40

Question 42

An insurance company issues N one-year insurance policies to lives aged 58. The policy has a death benefit of \$200,000, and is purchased with a single premium in advance. The policies are priced using the model with $q_{58} = 0.00032$. However, in fact q_{58} depends on various factors, and has the following distribution:

q_{58}	Probability
0.00028	0.26
0.00032	0.58
0.00033	0.14
0.00077	0.02

Calculate the variance of the present value of future loss on these polices.

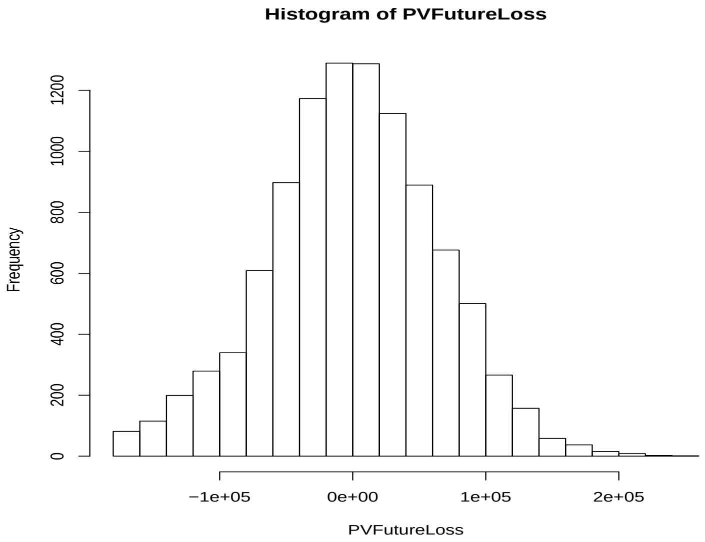
Question 43

A deferred annuity policy is sold to a life aged 47, paid for by level annual premiums in advance of \$15,184.40 until age 65. After age 65, it pays a life annuity of \$26,000 per year. Mortality follows a Gompertz law with $B = 0.0000164$ and $C = 1.088$. The policy pays a death benefit of \$100,000 during the deferment period. During the deferment period the interest rate is 4% . After the deferment period the yield curve is flat, with interest log-normally distributed with $\mu = \log(0.04)$ and $\sigma = 0.4$. You generate values from a $U(0, 1)$ distribution:

$$u_1 = 0.6129116, u_2 = 0.6120158, u_3 = 0.9504287$$
$$v_1 = 0.4396716, v_2 = 0.2549458, v_3 = 0.8275097$$

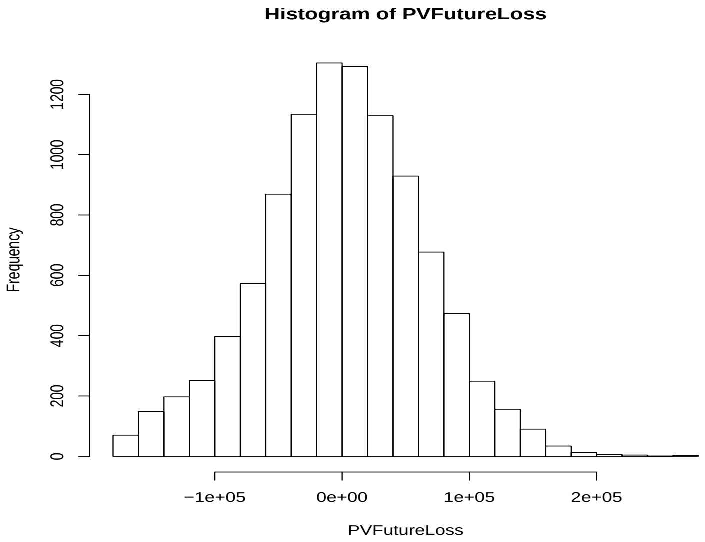
Using u_i to simulate future lifetime, and v_i to simulate future interest rates, obtain 3 samples from the distribution of the present value of future loss.

Simulated PV of Future Loss for Question 43



EPV future loss = 0.

Another Simulated PV of Future Loss for Question 43



EPV future loss = 176.42.

Question 44

In the second simulation, there were 10,000 simulated values. The mean of the simulated Present Value of future loss random variables was 176.4161, and the standard deviation was 64278.49. Calculate a 95% confidence interval for the true EPV of the loss on the policy.

Question 45

An insurance company sells a 10-year annual life insurance policy to a life aged 34, for whom the lifetable below is appropriate. The interest rate is $i = 0.04$. The death benefits are $180,000. The initial expenses are $300 plus 20% of the first premium. The renewal costs are 4% of each annual premium.

x	l_x	d_x
34	10000.00	3.13
35	9996.87	3.29
36	9993.58	3.47
37	9990.10	3.67
38	9986.44	3.88

x	l_x	d_x
39	9982.56	4.11
40	9978.45	4.36
41	9974.10	4.62
42	9969.47	4.92
43	9964.55	5.23

Calculate the cashflows associated with the policy if the annual premium is $90.

Answer to Question 45

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		160			−160.00
1	90	0.0	3.60	56.34	37.26
2	90	3.6	3.46	59.24	30.62
3	90	3.6	3.46	62.50	27.36
4	90	3.6	3.46	66.13	23.73
5	90	3.6	3.46	69.93	19.93
6	90	3.6	3.46	74.11	15.75
7	90	3.6	3.46	78.65	11.21
8	90	3.6	3.46	83.38	6.48
9	90	3.6	3.46	88.83	1.03
10	90	3.6	3.46	94.47	−4.61

Question 46

Repeat Question 45 including a reserve, where the reserve is the net premium reserve, calculated on the reserve basis $i = 0.03$, and mortality higher than in the table by a constant rate 0.004. This gives the following reserves: Premium=$767.4278.

t	${}_tV$
0	0
1	15.89511
2	29.38556
3	40.07908
4	47.51575

t	${}_tV$
5	51.39936
6	51.24223
7	46.53873
8	36.94503
9	21.56219

Answer to Question 46

t	${}_{t-1}V$	P	E_t	I	Death Benefits	${}_tVp_{34+t-1}$	Net Profit
0			160				-160.00
1	0.00	90	0.0	3.60	56.34	15.90	21.36
2	15.90	90	3.6	4.09	59.24	29.39	17.76
3	29.39	90	3.6	4.63	62.50	40.08	17.84
4	40.08	90	3.6	5.18	66.13	47.52	18.01
5	47.52	90	3.6	5.36	69.93	51.40	17.95
6	51.40	90	3.6	5.51	74.11	51.24	17.96
7	51.24	90	3.6	5.51	78.65	46.54	17.96
8	46.54	90	3.6	5.32	83.38	36.95	17.93
9	36.95	90	3.6	4.93	88.83	21.56	17.89
10	21.56	90	3.6	4.32	94.47	0.00	17.81

Profit Signatures

- The above profits can be calculated using one of the formulae:

$$\mathrm{Pr}_t = ({}_{t-1}V + P_t - E_t)(1+i) - S_t q_{x+t-1} - {}_tV p_{x+t-1}$$
$$\mathrm{Pr}_t = (P_t - E_t)(1+i) - S_t q_{x+t-1} - \Delta_t V$$

  where $\Delta_t V = {}_{t-1}V(1+i) - {}_tV p_{x+t-1}$ is the change in reserve.
- The final column Pr_t is called the profit vector of the contract. Pr_t is the expected end-of-year profit conditional on the contract still being in force at time $t-1$.
- The profit signature Π_t is the expected profit realised at time t, given by $\Pi_0 = \mathrm{Pr}_0$ and $\Pi_t = \mathrm{Pr}_t\, {}_{t-1}p_x$ for $t > 0$.
- We can then apply various profit measures to the profit signature to determine how profitable the contract is.

Question 47

Calculate the profit signatures for the contract in Question 45, both for the original case and the case (Question 46) with reserves.

Answer to Question 47

t	Without Reserves	With Reserves
0	−160.00	−160.00
1	37.26	21.36
2	30.61	17.75
3	27.34	17.83
4	23.71	17.99
5	19.90	17.93
6	15.72	17.93
7	11.19	17.92
8	6.46	17.88
9	1.03	17.84
10	−4.59	17.75

Notes on Profit Testing

- Easy to adapt this to Multiple Decrement Models.
- Profit testing is usually applied to a portfolio of policies, rather than a single policy.
- We have replaced random variables by their expected values. This is called deterministic profit testing.
- The profit signature is used to assess profitability. The profit vector is used for policies already in force.
- We will cover stochastic profit testing and profit testing for multiple decrement models later.

12.5 Profit Measures

Profit Measures

Net Present Value	Present value of profit signature at risk discount rate
Profit Margin	NPV as a proportion of EPV of premiums received
Partial NPV	$\mathrm{NPV}(t)$ is the NPV of all cash-flows up to time t
Internal Rate of Return	Interest rate at which NPV is zero
Discounted Payback Period	First time at which partial NPV is at least 0

Question 48

Calculate these profit measures for the policy in Question 45, both with and without reserves. Use risk discount rates of 1%, 5%, and 10% where appropriate. The profit signatures are recalled below:

t	Without Reserves	With Reserves
0	−160.00	−160.00
1	37.26	21.36
2	30.61	17.75
3	27.34	17.83
4	23.71	17.99
5	19.90	17.93
6	15.72	17.93
7	11.19	17.92
8	6.46	17.88
9	1.03	17.84
10	−4.59	17.75

Answer to Question 48

discount rate	Profit Measure	No Reserves	Reserves
1%	NPV	3.151168	12.69993
	Profit Margin	0.003666031	0.01477495
	Partial NPV(5)	-24.8471	-69.79779
	DPP	7 years	10 years
5%	NPV	-16.13285	-18.69238
	Profit Margin	-0.02214158	-0.02565442
	Partial NPV(5)	-38.03435	-79.3061
	DPP		
10%	NPV	-35.44164	-47.02866
	Profit Margin	-0.0583403	-0.07741364
	Partial NPV(5)	-51.73822	-89.09592
	DPP		
	IRR	1.60%	2.48%

Question 49

For the policy in Question 45, calculate the premium that achieves a risk discount rate of 10%.

Answer to Question 49

t	Premium (at $t-1$)	Expenses	Interest	Death Benefits	Net Cash Flow
0		160			-160.00
1	P	0.0	$0.04P$	56.34	$1.04P - 56.34$
2	P	$0.04P$	$0.0396P$	59.24	$0.9996P - 59.24$
3	P	$0.04P$	$0.0396P$	62.50	$0.9996P - 62.50$
4	P	$0.04P$	$0.0396P$	66.13	$0.9996P - 66.13$
5	P	$0.04P$	$0.0396P$	69.93	$0.9996P - 69.93$
6	P	$0.04P$	$0.0396P$	74.11	$0.9996P - 74.11$
7	P	$0.04P$	$0.0396P$	78.65	$0.9996P - 78.65$
8	P	$0.04P$	$0.0396P$	83.38	$0.9996P - 83.38$
9	P	$0.04P$	$0.0396P$	88.83	$0.9996P - 88.83$
10	P	$0.04P$	$0.0396P$	94.47	$0.9996P - 94.47$

Question 50

Calculate the reserves for the policy in Question 45 so that no year has a negative cash flow.

12.7 Using the Profit Test to Calculate Reserves

Recall that for Question 45, we calculated the following cash-flows.

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		160			-160.00
1	90	0.0	3.60	56.34	37.26
2	90	3.6	3.46	59.24	30.62
3	90	3.6	3.46	62.50	27.36
4	90	3.6	3.46	66.13	23.73
5	90	3.6	3.46	69.93	19.93
6	90	3.6	3.46	74.11	15.75
7	90	3.6	3.46	78.65	11.21
8	90	3.6	3.46	83.38	6.48
9	90	3.6	3.46	88.83	1.03
10	90	3.6	3.46	94.47	-4.61

Question 51

Recall Question 6, where a life insurance company sells a 10-year term disability income policy to a life aged 37. The transition intensities are

$$\mu_x^{01} = 0.0003 + 0.000002x$$
$$\mu_x^{10} = 0.00003 + 0.000001x$$
$$\mu_x^{02} = 0.0001 + 0.000001x^2$$
$$\mu_x^{12} = 0.0002 + 0.000002x$$

Premiums are payable annually in advance while healthy. Benefits of \$80,000 per year in arrear are payable if the life is sick at the end of a given year. A death benefit of \$200,000 is payable at the end of the year of death. The net annual premium for this policy using $i = 0.06$ is \$489.45. Use a profit test to calculate the reserves and the internal rate of return of the policy if the interest rate earned by the company is $i = 0.07$.

Answer to Question 51

t	p_{37+t}^{01}	p_{37+t}^{02}	p_{37+t}^{10}	p_{37+t}^{12}
0	0.0003746	0.0015050	0.0000674	0.0002750
1	0.0003766	0.0015808	0.0000684	0.0002770
2	0.0003785	0.0016587	0.0000694	0.0002790
3	0.0003805	0.0017385	0.0000704	0.0002810
4	0.0003825	0.0018204	0.0000714	0.0002830
5	0.0003845	0.0019042	0.0000724	0.0002850
6	0.0003865	0.0019900	0.0000734	0.0002870
7	0.0003884	0.0020778	0.0000744	0.0002890
8	0.0003904	0.0021676	0.0000754	0.0002910
9	0.0003924	0.0022594	0.0000764	0.0002930

Answer to Question 51 — Profit Vector in Healthy state

t	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Net Cash Flow
0		200				−200
1	489.45		34.26	29.96666	300.9938	192.75108
2	489.45		34.26	30.12527	316.1673	177.41896
3	489.45		34.26	30.28383	331.7389	161.68881
4	489.45		34.26	30.44234	347.7084	145.56071
5	489.45		34.26	30.60081	364.0759	129.03478
6	489.45		34.26	30.75924	380.8412	112.11110
7	489.45		34.26	30.91762	398.0041	94.78978
8	489.45		34.26	31.07596	415.5646	77.07092
9	489.45		34.26	31.23425	433.5226	58.95464
10	489.45		34.26	31.39249	451.8780	40.44103

Answer to Question 51 — Profit Vector in Sick state

t	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Net Cash Flow
0		200				−200
1	489.45		34.26	79972.61	55.00074	−79503.89
2	489.45		34.26	79972.37	55.40126	−79504.06
3	489.45		34.26	79972.13	55.80181	−79504.22
4	489.45		34.26	79971.89	56.20238	−79504.38
5	489.45		34.26	79971.65	56.60299	−79504.54
6	489.45		34.26	79971.41	57.00362	−79504.70
7	489.45		34.26	79971.17	57.40429	−79504.86
8	489.45		34.26	79970.93	57.80498	−79505.02
9	489.45		34.26	79970.69	58.20571	−79505.18
10	489.45		34.26	79970.45	58.60646	−79505.34

Answer to Question 51 — Reserves in Sick state

t	Reserve	Prem.	Interest	Exp. Disability Ben.	Exp. Death Ben.	Exp. Reserve	Net Cash Flow
2	530748.32	489.45	37186.64	79972.37	55.40	488396.64	−567900.70
3	488566.86	489.45	34233.94	79972.13	55.80	443262.32	−522766.54
4	443418.14	489.45	31073.53	79971.89	56.20	394953.03	−474457.41
5	395093.05	489.45	27690.78	79971.65	56.60	343245.02	−422749.56
6	343367.74	489.45	24070.00	79971.41	57.00	269064.28	−348568.98
7	269161.28	489.45	18875.55	79971.17	57.40	208497.72	−288002.58
8	208573.51	489.45	14634.41	79970.93	57.80	143668.64	−223173.66
9	143721.30	489.45	10094.75	79970.69	58.21	74276.61	−153781.79
10	74276.61	489.45	5235.55	79970.45	58.61	0	−79505.34

Answer to Question 51 — Reserves in Healthy state

t	Res.	Prem.	Int.	Exp. Dis. Ben.	Exp. Death Ben.	Exp. Res. Sick	Exp. Res. Healthy	Net Cash Flow
1	21.65	489.45	34.26	29.97	300.99	198.81	17.11	−23.17
2	17.14	489.45	34.26	30.13	316.17	183.98	11.78	−18.34
3	11.80	489.45	34.26	30.28	331.74	167.85	6.47	−12.63
4	6.49	489.45	34.26	30.44	347.71	150.34	2.15	−6.94
5	2.16	489.45	34.26	30.60	364.08	131.34		−2.31
6		489.45	34.26	30.76	380.84	103.49		8.62
7		489.45	34.26	30.92	398.00	80.61		14.18
8		489.45	34.26	31.08	415.56	55.83		21.24
9		489.45	34.26	31.23	433.52	29.00		29.95
10		489.45	34.26	31.39	451.88			40.44

Risks with Whole-life and long-term endowment insurance

For the Policyholder:

- Policyholder's needs may change over time.
- Market conditions may mean the investment component is not sufficient to cover future needs.

For the Insurer:

- Term of investment can be very long. Fixed Interest rates for this period may not be available.
- If the insurance company is too conservative in the interest rate it offers, the investment component won't be attractive to investors.

Solution — Participating (or with profit) Insurance

- Premiums are set on a very conservative basis.
- Emerging surplus is shared with policyholders.
- Variety of options for sharing surplus with policyholders:
    - Cash dividends
    - Reductions in premium
    - Increase in benefits (reversionary bonus)

13.4 Universal Life Insurance

Details

- Credited interest added to policyholders (notional) account.
- Rate may be based on published rates, or subject to a minimum.
- Regular withdrawls from account to cover cost of insurance.
- Death benefit: account value plus Additional Death Benefit (ADB).
- ADB subject to corridor factor requirement.
- Type A has level total death benefit. Type B has level ADB.
- Premiums subject to minimum level and term, otherwise flexible.
- Expense charges (% premium/account) deducted from account.
- Cost of Insurance charges regularly deducted from account.
- Surrender charge may apply, to cover acquisition expenses.
- No-lapse guarantee allows policyholder to retain coverage by paying a minimum premium, even if account value is zero.

Changes to Account Value

$$AV_t = (AV_{t-1} + P_t - EC_t - CoI_t)(1 + i_t)$$

where

- AV_t is the account value at time t.
- P_t is the premium at the start of year t.
- EC_t is the expense charge at the start of year t.
- CoI_t is the cost of insurance at the start of year t.

Profit Testing Universal Life Policies

- Project account values assuming policy remains in force.
- $ADB_t = DB_t - AV_t$.
- $CoI_t = q^*_{x+t-1} \nu_q ADB_t$.

Type A Policies

Death benefit is constant (subject to corridor factor requirement). As account value increases, additional death benefit decreases.

Type B Policies

Additional death benefit is constant (subject to corridor factor requirement). As account value increases, so does total death benefit.

Question 52

Type B universal life insurance policy, sold to an individual aged 53. Initial annual premium \$3,160. ADB \$250,000. Mortality:

x	l_x	d_x	x	l_x	d_x	x	l_x	d_x
53	10000.00	10.92	58	9933.66	17.59	63	9824.19	29.79
54	9989.08	11.95	59	9916.06	19.49	64	9794.40	33.23
55	9977.12	13.12	60	9896.58	21.62	65	9761.16	37.10
56	9964.01	14.43	61	9874.96	24.03	66	9724.06	41.46
57	9949.57	15.92	62	9850.93	26.74	67	9682.60	46.34

- Policyholder pays premium \$3,160 for 8 years, then stops.
- The credited interest rate is $i = 0.04$.
- CoI based on 110% mortality in the above table, and $i = 0.03$.
- Expense charges 1.5% of account value (after premium is paid).

Project the account value for the next 15 years.

Answer to Question 52

AV_{t-1}	P_t	EC_t	CoI_t	interest	AV_t
0.00	3,160	47.40	291.55	112.84	2,933.89
2,933.89	3,160	91.41	319.40	227.32	5,939.37
5,939.37	3,160	136.49	351.09	344.47	8,956.26
8,956.26	3,160	181.74	386.66	461.91	12,009.77
12,009.77	3,160	227.55	427.20	580.60	15,095.62
15,095.62	3,160	273.83	472.77	700.36	18,209.37
18,209.37	3,160	320.54	524.77	820.96	21,345.02
21,345.02	3,160	367.58	583.27	942.17	24,496.34
24,496.34	0	367.45	649.70	939.17	24,418.36
24,418.36	0	366.28	724.74	933.09	24,260.44
24,260.44	0	363.91	809.60	923.48	24,010.41
24,010.41	0	360.16	905.83	909.78	23,654.20
23,654.20	0	354.81	1,014.77	891.38	23,176.00
23,176.00	0	347.64	1,138.35	867.60	22,557.61
22,557.61	0	338.36	1,278.34	837.64	21,778.54

Question 53

Profit test the policy from Question 52. Calculate the profit margin under the following assumptions.

- The insurer earns interest rate at $i = 0.05$.
- Mortality is as shown in the lifetable.
- Expenses are: initial — $1,800; renewal — 1% of premium.
- The surrender charge and surrender rates are:

Year	charge	rate
1	$3,000	5%
2	$2,500	5%
3	$2,000	2%
4	$1,600	2%

Year	charge	rate
5	$1,200	2%
6	$900	2%
7	$500	3%
8–12	$0	3%
13-14	$0	4%

- The risk discount rate is $i = 0.1$.

Answer to Question 53

AV_{t-1}	P_t	E_t	I_t	EDB_t	ESB_t	EAV_t	Pr_t
0	0	1800.00					−1800.00
0.00	3160	0.00	158.00	276.20	0.00	2784.15	257.64
2933.89	3160	31.60	303.11	306.18	171.76	5635.65	251.81
5939.37	3160	31.60	453.39	340.53	138.94	8765.59	276.09
8956.26	3160	31.60	604.23	379.45	207.89	11752.53	349.02
12009.77	3160	31.60	756.91	424.17	277.47	14770.04	423.40
15095.62	3160	31.60	911.20	474.93	345.57	17813.58	501.13
18209.37	3160	31.60	1066.89	533.33	624.12	20663.97	583.23
21345.02	3160	31.60	1223.67	599.66	733.28	23709.54	654.60
24496.34	0	0.00	1224.82	667.78	730.77	23628.17	694.44
24418.36	0	0.00	1220.92	744.47	725.84	23468.75	700.22
24260.44	0	0.00	1213.02	830.88	718.13	23219.48	704.97
24010.41	0	0.00	1200.52	928.44	707.22	22866.73	708.54
23654.20	0	0.00	1182.71	1038.28	923.52	22164.40	710.72
23176.00	0	0.00	1158.80	1162.09	898.46	21562.98	711.28
22557.61	0	0.00	1127.88	1301.27	21674.27	0.00	709.96

Question 54

Type A universal life insurance policy, sold to an individual aged 53. Initial annual premium $5,180. DB $400,000. Mortality:

x	l_x	d_x	x	l_x	d_x	x	l_x	d_x
53	10000.00	10.92	57	9949.57	15.92	61	9874.96	24.03
54	9989.08	11.95	58	9933.66	17.59	62	9850.93	26.74
55	9977.12	13.12	59	9916.06	19.49	63	9824.19	29.79
56	9964.01	14.43	60	9896.58	21.62	64	9794.40	33.23

- Policyholder pays premium $5,180 for 12 years.
- The credited interest rate is $i = 0.05$.
- CoI based on mortality in the above table, and $i = 0.06$.
- Expense charges 1% of account value (after premium is paid).

Project the account value for the next 12 years.

Answer to Question 54

AV_{t-1}	P_t	E_t	Col_t	interest	AV_t
0.00	5160	51.60	407.25	235.06	4936.20
4936.20	5160	100.96	440.68	477.73	10032.29
10032.29	5160	151.92	478.19	728.11	15290.28
15290.28	5160	204.50	519.58	986.31	20712.51
20712.51	5160	258.73	566.03	1252.39	26300.14
26300.14	5160	314.60	617.26	1526.41	32054.69
32054.69	5160	372.15	674.70	1808.39	37976.24
37976.24	5160	431.36	737.96	2098.35	44065.26
44065.26	5160	492.25	808.35	2396.23	50320.89
50320.89	5160	554.81	886.05	2702.00	56742.03
56742.03	5160	619.02	971.88	3015.56	63326.69
63326.69	5160	684.87	1066.86	3336.75	70071.72

Question 55

Profit test the policy from Question 54. Calculate the profit margin under the following assumptions.

- The insurer earns interest rate at $i = 0.08$.
- Mortality is as shown in the lifetable.
- Initial expenses are $2,800. Renewal expenses are 1.5% of premium paid.
- The surrender charge and surrender rates are:

Year	charge	rate
1	$3,300	5%
2	$2,600	5%
3	$1,800	2%
4	$1,200	2%

Year	charge	rate
5	$800	2%
6	$300	2%
7-11	$ 0	3%

- The risk discount rate is $i = 0.15$.

Answer to Question 55

AV_{t-1}	P_t	E_t	I_t	EDB_t	ESB_t	EAV_t	Pr_t
0	0	2800.00					−2800.00
0.00	5160	77.40	406.61	436.80	81.72	4684.27	286.42
4936.20	5160	77.40	801.50	478.52	371.17	9519.27	451.34
10032.29	5160	77.40	1209.19	526.00	269.45	14964.77	563.86
15290.28	5160	77.40	1629.83	579.28	389.69	20268.86	764.88
20712.51	5160	77.40	2063.61	640.03	509.19	25732.90	976.61
26300.14	5160	77.40	2510.62	708.30	633.97	31357.97	1193.12
32054.69	5160	77.40	2970.98	786.20	1137.05	36764.55	1420.48
37976.24	5160	77.40	3444.71	873.84	1319.07	42649.93	1660.71
44065.26	5160	77.40	3931.83	973.37	1505.95	48692.49	1907.88
50320.89	5160	77.40	4432.28	1085.79	1697.64	54890.37	2161.98
56742.03	5160	77.40	4945.97	1212.92	1894.04	61240.62	2423.01
63326.69	5160	77.40	5472.74	1357.10	69833.98	0.00	2690.95

Question 56

A life age 48 has a Type A Universal life insurance policy that has been in effect for 7 years.

- The current account value is $107,389.
- The annual premium is $16,000.
- The Expense charge is 1% of account value (after premium).
- The Credited interest rate is $i = 0.07$.
- The total death benefit is $300,000, and the corridor factor requirement is 2.3.
- The mortality rate is $q_{47} = 0.000265$.
- The insurance is priced using an interest rate of $i = 0.06$.

Calculate the Cost of Insurance charge for the year.

Other features of Universal Life Insurance

- Policy may include additional guarantees. (e.g. no-lapse guarantee).
- This may require additional reserves above the policy value.
- It might sometimes be possible to hold smaller reserve than account value, because surrender value is less than account value. This is risky because surrenders are difficult to predict.
- Often profit test needs to be repeated with a range of different assumptions on credited interest rates, rates of return, etc. including the effects of guaranteed values.

13.5 Comparison of UL and Whole Life Insurance Policies

Differences between UL and whole life or endowment policies

- UL policies generally have better surrender values. Whole life policies may offer better death benefits.
- UL policies have less certainty in the value of benefits. This is similar to participating policies with reversionary bonuses.
- UL policies have more flexibility on payment of premiums.

14.2 Equity-Linked Insurance

Notes

- Also called unit-linked insurance (UK, Europe), variable annuities (USA), and segregated funds (Canada).
- Policyholder pays a regular premium. Premiums accumulate in the policyholder's fund.
- Management Charges deducted regularly, and transfered to insurer's fund, which covers expenses and insurance payments.
- Maturity and death benefits based on policyholder's fund. For example, death benefit might be 110% of policyholder's fund.
- There may be a guaranteed minimum maturity benefit (GMMB) or a guaranteed minimum death benefit (GMDB).
- Bid-offer spread is the percentage of the premium which is allocated to the insurance fund rather than the policyholder's fund.
- Allocation percentage is the percentage of the premium after the bid-offer spread, allocated to the policyholder's fund.

14.3 Deterministic Profit Testing for Equity-Linked Insurance

Question 57

Consider the following equity-linked insurance policy:

- Annual premiums: $6,000.
- expenses: 4% 1st premium, 1% subsequent premiums.
- Year-end management expense: 0.5% of fund value.
- Year-end death benefit: 120% of fund value.
- Surrenders: fund value.
- GMMB: total of premiums.
- Surrender rate: 2% per year
- $q_x = 0.0004 + 0.00002x$
- Annual return: 8%
- Initial expenses: $200 plus 25% of first premium
- Renewal expenses: 0.4% of subsequent premiums.

Calculate the profit signature for a 10-year policy sold to a life aged 48, with no reserves:

Answer to Question 57

t	Alloc. Prem.	Start Value	Int.	Fund before	Mgmt. Charge	Fund
1	5760	0.00	460.80	6220.80	31.10	6189.70
2	5940	6189.70	970.33	13099.43	65.50	13033.93
3	5940	13033.93	1517.91	20491.84	102.46	20389.39
4	5940	20389.39	2106.35	28435.74	142.18	28293.56
5	5940	28293.56	2738.69	36972.24	184.86	36787.38
6	5940	36787.38	3418.19	46145.57	230.73	45914.84
7	5940	45914.84	4148.38	56003.23	280.02	55723.21
8	5940	55723.21	4933.06	66596.27	332.98	66263.29
9	5940	66263.29	5776.26	77979.55	389.90	77589.66
10	5940	77589.66	6682.3728	90212.03	451.06	89760.97

Answer to Question 57

t	Unalloc. Prem.	Exp.	Int.	Mgmt. Charge	Death Benefit	$\Pr_t$
0	0	1700	0	0	0	-1700.00
1	240	0	19.20	31.10	1.68	288.62
2	60	24	2.88	65.50	3.60	100.78
3	60	24	2.88	102.46	5.71	135.63
4	60	24	2.88	142.18	8.04	173.02
5	60	24	2.88	184.86	10.59	213.15
6	60	24	2.88	230.73	13.41	256.20
7	60	24	2.88	280.02	16.49	302.41
8	60	24	2.88	332.98	19.88	351.98
9	60	24	2.88	389.90	23.59	405.19
10	60	24	2.88	451.06	27.65	462.29

Question 58

Recalculate the profit signature for the policy from Question 57 under the assumption that the investment funds return 0.5% per year.

Answer to Question 58

t	Alloc. Prem.	Start Value	Int.	Fund before	Mgmt. Charge	Fund
1	5760	0.00	28.80	5788.80	28.44	5760.36
2	5940	5760.36	58.50	11758.86	58.79	11700.07
3	5940	11700.07	88.20	17728.27	88.64	17639.63
4	5940	17639.63	117.90	23697.53	118.49	23579.04
5	5940	23579.04	147.60	29666.64	148.33	29518.30
6	5940	29518.30	177.29	35635.59	178.18	35457.41
7	5940	35457.41	206.99	41604.40	208.02	41396.38
8	5940	41396.38	236.68	47573.06	237.87	47335.20
9	5940	47335.20	266.38	53541.58	267.71	53273.87
10	5940	53273.87	296.07	59509.94	297.55	59212.39

Answer to Question 58

t	Unalloc. Prem.	Exp.	Int.	Mgmt. Charge	Death Benefit	GMMB	$\Pr_t$
0	0	1700	0	0	0		−1700
1	240	0	1.20	28.44	1.57		268.07
2	60	24	0.18	58.79	3.23		91.74
3	60	24	0.18	88.64	4.94		119.88
4	60	24	0.18	118.49	6.70		147.97
5	60	24	0.18	148.33	8.50		176.01
6	60	24	0.18	178.18	10.35		204.01
7	60	24	0.18	208.02	12.25		231.95
8	60	24	0.18	237.87	14.20		259.85
9	60	24	0.18	267.71	16.20		287.69
10	60	24	0.18	297.55	18.24	787.61	−472.12

14.4 Stochastic Profit Testing

Stochastic Profit Testing

- Investment returns are not deterministic, and importantly, are not diversifiable.
- Taking the average results does not adequately describe the range of outcomes possible, and the inherent risks.
- Complicated benefits like GMMB are not linear functions of the rate of return, so average returns won't give average costs. [GMMB and GMDB are a form of option. Pricing of options is beyond the scope of this course, and is covered in MATH 3900.]
- Instead of using a single profit test, we simulate a large number of different returns, and calculate the profit for each one.
- For the annual policy, we simulate the annual investment returns usually following a log-normal distribution. We use the simulated returns in place of the deterministic returns in the previous examples.

Question 59

For the Policy in Question 57, with an additional GMDB equal to 120% of the total of all premiums paid, perform a stochastic profit test using a log-normal distribution for the rate of return with $\mu = 0.08$ and $\sigma = 0.09$. Use the following values from a uniform distribution on $[0, 1]$ for the simulation:

0.5388720	0.2815602	0.1209265	0.8930640
0.5237917	0.3144833	0.8926775	0.2738433
0.1899877	0.1755291		

Answer to Question 59

t	Alloc. Prem.	Start Value	Rate	Int.	Fund before	Mgmt. Charge	Fund
1	5760	0.00	9.3%	534.78	6294.78	31.47	6263.31
2	5940	6263.31	2.8%	346.03	12549.34	62.75	12486.59
3	5940	12486.59	−2.5%	−460.94	17965.65	89.83	17875.82
4	5940	17875.82	21.2%	5037.33	28853.15	144.27	28708.88
5	5940	28708.88	8.9%	3087.93	37736.81	188.68	37548.12
6	5940	37548.12	3.7%	1617.26	45105.38	225.53	44879.85
7	5940	44879.85	21.1%	10737.39	61557.24	307.79	61249.46
8	5940	61249.46	2.6%	1762.20	68951.66	344.76	68606.90
9	5940	68606.90	0.1%	73.48	74620.38	373.10	74247.28
10	5940	74247.28	−0.4%	−314.40	79872.89	399.36	79473.52

Answer to Question 59

t	Unalloc. Prem.	Exp.	Int.	Mgmt. Charge	Death Benefit	$\Pr_t$
0	0	1700	0	0	0	-1700
1	240	0	22.28	31.47	10.22	283.53
2	60	24	1.02	62.75	20.68	79.09
3	60	24	-0.90	89.83	30.24	94.69
4	60	24	7.61	144.27	48.92	138.96
5	60	24	3.21	188.68	64.88	163.01
6	60	24	1.34	225.53	78.63	184.24
7	60	24	7.61	307.79	108.78	242.61
8	60	24	0.94	344.76	123.49	258.21
9	60	24	0.04	373.10	135.43	273.71
10	60	24	-0.14	399.36	146.87	288.36

Question 60

For the same analysis as Question 59, simulate 5000 sets of investment returns to estimate the distribution of the NPV at discount rate 4% for the policy.

R Code for Question 60

Simulate random returns:

```
U<-runif(50000)
dim(U)<-c(5000,10)
R<-exp(qnorm(U)*0.09+0.08)
```

Calculate survival probabilities:

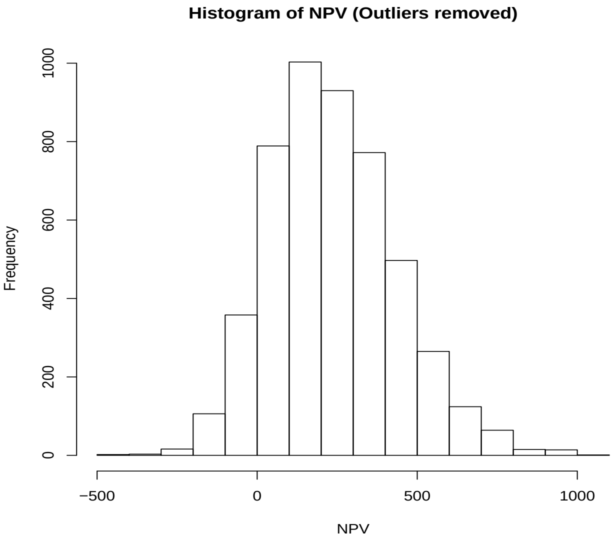
```
NPV<-1
length(NPV)<-0
px<-c(1,0.97864-0.00002*(0:8))
S<-cumprod(px)
```

R Code for Question 60

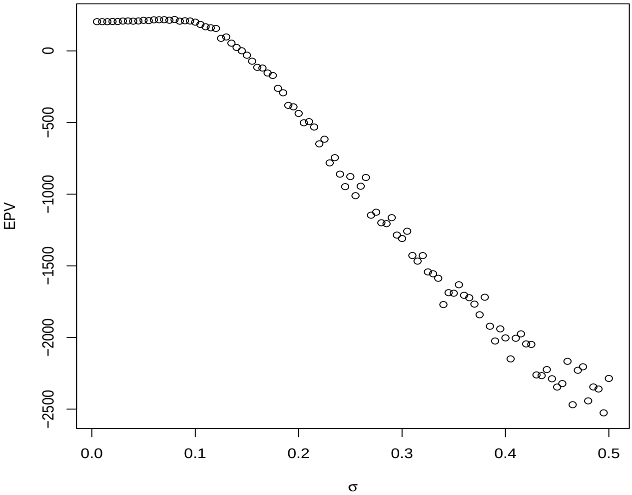
Calculate NPV for simulated results

```
for(i in 1:5000){
  X<-5760*c(R[i,1]/200,R[i,1]*199/200)
  dim(X)<-c(1,2)
  for(j in 2:10){
    stval<-(5940+X[(j-1),2])
    X<-rbind(X,stval*c(R[i,j]/200,R[i,j]*199/200))
  }
  DB<-1.2*pmax(X[,2],6000*(1:10))-X[,2]
  EDB<-DB*(0.0004+0.00002*(48:57))
  Pr<-36*R[i,2:10]+X[2:10,1]-EDB[2:10]
  Pr<-c(240*R[i,1]+X[1,1]-EDB[1],Pr)
  NPV<-c(NPV,sum(Pr*S/1.04^(1:10))-1700)
}
```

Histogram of NPV at 4% for Question 60



Plot of EPV versus σ for Question 60



14.5 Stochastic Pricing

Stochastic Pricing

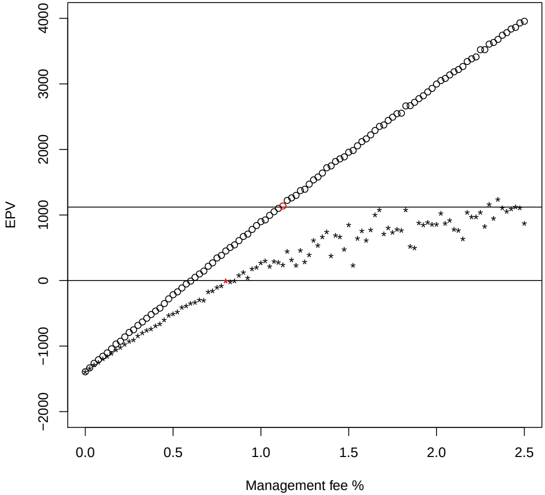
- Expected value premium principle can involve large risk. For diversifiable risks, this is manageable. For non-diversifiable risks, this is dangerous.
- Alternative approach is to set conditions — for example, probability of loss less than 5%, and EPV of profit at least 60% of acquisition costs.
- Using stochastic profit testing, we can test whether a particular contract satisfies these criteria.
- Since benefits and acquisition costs increase with the premium, changing the premium has little effect on the profitability of the contract.
- Alternative changes include changing the management charge, expense deductions or minimum benefits.

Question 61

Use Monte Carlo simulation to estimate the Management fee in the policy from Question 59 which satisfies the following criteria, using a 10% risk discount rate:

- Less than 5% probability of net loss.
- Expected NPV at least 70% of acquisition costs.

EPV and Quantiles vs. Management Fee (Question 61)



Question 62

An equity-linked insurance policy has the following properties:

- Annual premiums \$11,000.
- EC 10% first prem and 1% subsequent prem.
- Year-end MF 1% fund val
- Year-end DB 150% fund val
- Surrenders — full fund val
- GMMB total of premiums.
- Init. exp. \$400 + 10% prem
- Renew exp. 0.5% prem
- $q_x = 0.0002 + 0.00003x$.
- sold to life aged 44.
- matures in 10 years.
- Surrenders 2% per year.
- insurers' annual return 5%.
- Annual returns log-normal $\mu = 0.05, \sigma = 0.09$.

The insurance company simulates 5000 sets of annual returns. The expected fund value at the end of each year (after the management charge) is given in the following table.

Question 62(Continued)

Year	Expected fund value
1	$10,777.76
2	$23,738.14
3	$36,545.19
4	$50,891.99
5	$69,002.40

Year	Expected fund value
6	$87,803.32
7	$105,490.06
8	$126,203.20
9	$145,938.26
10	$168,093.55

In 233 of their simulations, the fund value at the end of year 10 was less than $110,000. The mean fund value at the end of year 10 for these simulations was $89,492.45. The policy has no reserves. Calculate the NPV of this policy for the simulated returns at a risk discount rate of 10%.

Question 63

A life insurance company is using simulation to determine the GMMB, expense charges and management charge for an equity-linked insurance policy. They plan to arrange these so that the NPV of the policy at a risk discount rate of 10% is at least 50% of the acquisition costs, and the probability of making a loss at the risk discount rate is at most 2%. The insurance company simulates 10000 sets of investment returns.

The company finds that even with the management charge increasing to 100%, the probability of making a loss is still more than 2%. Why does this happen, and what should they do to solve this problem?

Question 63

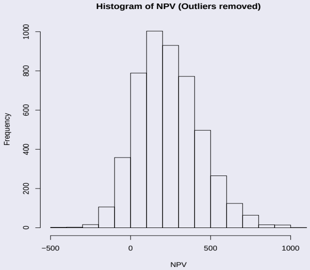
A life insurance company is using simulation to determine the GMMB, expense charges and management charge for an equity-linked insurance policy. They plan to arrange these so that the NPV of the policy at a risk discount rate of 10% is at least 50% of the acquisition costs, and the probability of making a loss at the risk discount rate is at most 2%. The insurance company simulates 10000 sets of investment returns.

The company finds that even with the management charge increasing to 100%, the probability of making a loss is still more than 2%. Why does this happen, and what should they do to solve this problem?

(i) Increase the expense charges
(ii) Decrease the expense charges
(iii) Increase the GMMB
(iv) Decrease the GMMB

Question 64

For the policy in Question 59, assume that investment gains are log-normally distributed with $\mu = 0.08$, and $\sigma = 0.12$.



(a) Calculate the 98% quantile reserve at time 0, based on 5000 simulations, with reserves earning an interest rate $i = 0.04$.
(b) Calculate the 98% TCE reserve at time 0, based on 5000 simulations, with reserves earning an interest rate $i = 0.04$.

Question 65

For the policy in Question 64, suppose the company uses the 98% quantile reserve, as in part (a), and updates the reserve each year using the same procedure (with interest rate 4%. Suppose the annual returns are as follows:

Year	Return
1	-12.0294%
2	2.2582%
3	4.7540%
4	8.3983%
5	-6.6978%

Year	Return
6	24.9924%
7	31.7335%
8	0.7792%
9	2.5191%
10	8.4262%

Calculate the profit vector of the policy.

Answer to Question 65

t	Alloc. Prem.	Start Value	Rate	Int.	Fund before	Mgmt. Charge	Fund
1	5760	0.00	−12.03%	−692.89	5067.11	25.34	5041.77
2	5940	5041.77	2.26%	247.99	11229.76	56.15	11173.61
3	5940	11173.61	4.75%	813.59	17927.20	89.66	17837.57
4	5940	17837.57	8.40%	1996.92	25774.49	128.87	25645.62
5	5940	25645.62	−6.70%	−2115.54	29470.08	147.35	29322.73
6	5940	29322.73	24.99%	8812.99	44075.72	220.38	43855.34
7	5940	43855.34	31.73%	15801.80	65597.14	327.99	65269.15
8	5940	65269.15	0.78%	554.85	71764.01	358.82	71405.19
9	5940	71405.19	2.52%	1948.42	79293.61	396.47	78897.14
10	5940	78897.14	8.43%	7148.56	91985.70	459.93	91525.77

Answer to Question 65

t	Unalloc. Prem.	Exp.	Int.	Int. on Res.	Mgmt. Charge	Death Benefit	Change in Res	$\Pr_t$
0	0	1700	0.00	0.00	0.00	0.00	1051.77	−2751.77
1	240	0	−28.87	42.07	25.34	2.94	1024.68	−749.08
2	60	24	0.81	83.06	56.15	4.45	662.13	−490.57
3	60	24	1.71	109.54	89.66	5.27	428.83	−197.21
4	60	24	3.02	126.70	128.87	7.28	-846.29	1133.60
5	60	24	−2.41	92.85	147.35	9.62	2752.42	−2488.25
6	60	24	9.00	202.94	220.38	12.81	-3950.11	4405.62
7	60	24	11.42	44.94	327.99	19.32	-1123.45	1524.48
8	60	24	0.28	0.00	358.82	21.42	0	373.68
9	60	24	0.91	0.00	396.47	23.98	0	409.39
10	60	24	3.03	0.00	459.93	28.19	0	470.77

$$\text{NPV} = -1330.017$$