ACSC/STAT 4720, Life Contingencies II
Fall 2017
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Homework Sheet 3
Model Solutions

Basic Questions

1. *A life aged 42 wants to buy a 5-year term insurance policy. A life-table based on current-year mortality is:*

x	l_x	d_x
42	10000.00	6.78
43	9993.22	7.35
44	9985.87	7.98
45	9977.89	8.66
46	9969.24	9.39
47	9959.84	10.19

The insurance company uses a single-factor scale function $q(x,t) = q(x,0)(1-\phi_x)^t$ to model changes in mortality. The insurance company uses the following values for ϕ_x:

x	ϕ_x
42	0.01
43	0.03
44	0.02
45	0.025
46	0.015
47	0.02

Calculate $A^{1}_{42:\overline{5}|}$ at interest rate $i = 0.05$, taking into account the change in mortality.

We have

$$\begin{aligned}
A^{1}_{42:\overline{5}|} &= q(42,2017)(1.05)^{-1} + (1-q(42,2017))q(43,2018)(1.05)^{-2} + (1-q(42,2017))(1-q(43,2018))q(44,2019)(1.05)^{-3}\\
&\quad + (1-q(42,2017))(1-q(43,2018))(1-q(44,2019))q(45,2020)(1.05)^{-4}\\
&\quad + (1-q(42,2017))(1-q(43,2018))(1-q(44,2019))(1-q(45,2020))q(46,2021)(1.05)^{-5}\\
&= q_{42}(1.05)^{-1} + (1-q_{42})q_{43}(1-\phi_{43})(1.05)^{-2} + (1-q_{42})(1-q_{43}(1-\phi_{43}))q_{44}(1-\phi_{44})^2(1.05)^{-3}\\
&\quad + (1-q_{42})(1-q_{43}(1-\phi_{43}))(1-q_{44}(1-\phi_{44})^2)q_{45}(1-\phi_{45})^3(1.05)^{-4}\\
&\quad + (1-q_{42})(1-q_{43}(1-\phi_{43}))(1-q_{44}(1-\phi_{44})^2)(1-q_{45}(1-\phi_{45})^3)q_{46}(1-\phi_{46})^4(1.05)^{-5}\\
&= 0.00330747714529
\end{aligned}$$

2. *Using the lifetable from Question 1, the insurance company now uses the following mortality scale based on both age and year:*

			t			
x	*2017*	*2018*	*2019*	*2020*	*2021*	*2022*
42	0.01	0.015	0.015	0.02	0.02	0.015
43	0.03	0.03	0.025	0.02	0.015	0.02
44	0.02	0.03	0.03	0.025	0.02	0.015
45	0.025	0.03	0.025	0.015	0.015	0.02
46	0.015	0.02	0.015	0.01	0.015	0.01
47	0.02	0.015	0.01	0.015	0.02	0.025

Use this mortality scale to calculate $A^{1}_{42:\overline{5}|}$.

By multiplying $1-\phi(x,t)$, we get the following factors for $\frac{q(x,t)}{q_{x,2017}}$:

			t			
x	2017	2018	2019	2020	2021	2022
42	1	0.985	0.970225	0.9508205	0.93180409	0.91782702865
43	1	0.97	0.94575	0.926835	0.912932475	0.8946738255
44	1	0.97	0.9409	0.9173775	0.89902995	0.88554450075
45	1	0.97	0.94575	0.93156375	0.91759029375	0.899238487875
46	1	0.98	0.9653	0.955647	0.941312295	0.93189917205
47	1	0.985	0.97515	0.96052275	0.941312295	0.917779487625

This gives the following values for $q(x,t)$:

			t			
x	2017	2018	2019	2020	2021	2022
42	0.0006780000	0.00066783	0.00065781255	0.000644656299000	0.00063176317302	0.0006222867254
43	0.0007354987	0.000713433739	0.000695597895525	0.000681685937615	0.00067146064855	0.0006580314355
44	0.0007991292	0.000775155324	0.00075190066428	0.000733103147673	0.00071844108472	0.0007076644684
45	0.0008679190	0.00084188143	0.00082083439425	0.000808521878336	0.000796394050161	0.0007804661691
46	0.0009418973	0.000923059354	0.00090921346369	0.000900121329053	0.000886619509117	0.0008777533140
47	0.0010231088	0.001007762168	0.00099768454632	0.000982719278125	0.000963064892563	0.0009389882702

We therefore get

$$
\begin{aligned}
A^{1}_{42:\overline{5}|} &= 0.000678(1.05)^{-1} \\
&\quad + (1-0.000678)0.0007134337(1.05)^{-2} \\
&\quad + (1-0.000678)(1-0.0007134337)0.00075190066428(1.05)^{-3} \\
&\quad + (1-0.000678)(1-0.0007134337)(1-0.00075190066428)0.000808521878336(1.05)^{-4} \\
&\quad + (1-0.000678)(1-0.0007134337)(1-0.00075190066428)(1-0.000808521878336)0.000886619509117(1.05)^{-5} \\
&= 0.003297330
\end{aligned}
$$

3. *A pensions company has the current mortality scale for 2017:*

x	$\phi(x,2017)$	$\left.\frac{d\phi(x,t)}{dt}\right\|_{x,t=2017}$	$\left.\frac{d\phi(x+t,t)}{dt}\right\|_{x,t=2017}$
42	*0.028763796*	0.00254272963	−0.0010005971
43	*0.013387987*	−0.00007704268	−0.0015410004
44	*0.012284496*	0.00122050593	−0.0002926677
45	*0.020186718*	−0.00230931319	−0.0006144058
46	*0.023344489*	−0.00079030424	−0.0023352259
47	*0.007762005*	0.00227442877	0.0053024871

Current mortality is given in lifetable in Question 1. The company assumes that from 2030 onwards, we will have $\phi(x,t) = 0.015$ for all x and t. Calculate $A^{1}_{42:\overline{5}|}$ using the average of age-based and cohort-based effects.

We use cubic interpolation between 2017 and 2030. Suppose we have $f(t) = \phi(x, 2017+t) = at^3 + bt^2 + ct + d$, and we are given $f(0) = p$, $f'(0) = q$, $f(13) = 0.015$ and $f'(13) = 0$, we solve the following:

$$\begin{aligned} d &= p \\ c &= q \\ a \times 13^3 + b \times 13^2 + c \times 13 + d &= 0.015 \\ 3a \times 13^2 + 2b \times 13 + c &= 0 \\ 13^3 a - 13c - 2d &= -0.03 \\ a &= \frac{13c + 2d - 0.03}{13^3} \\ b &= \frac{0.045 - 26c - 3d}{13^2} \end{aligned}$$

This gives the following coefficients for all age-based effects:

x	a	b	c	d
42	0.00002757536512970	−0.0006355169134910	0.00254272963	0.028763796
43	−0.00000192334130178	0.0000404683353846	−0.00007704268	0.013387987
44	0.00000474991765589	−0.0001395659300590	0.00122050593	0.012284496
45	−0.00000894293831133	0.0002632070351480	−0.00230931319	0.020186718
46	0.00000291990117433	−0.0000265417559763	−0.00079030424	0.023344489
47	0.00000686917797451	−0.0002214270001180	0.00227442877	0.007762005

For cohort-based effects the coefficients are:

x	a	b	c	d
42	0.00000660893477469	−0.00009038972426040	−0.0010005971	0.028763796
43	−0.00001058581301780	0.00026569260000000	−0.0015410004	0.013387987
44	−0.00000420377246245	0.00009323001301780	−0.0002926677	0.012284496
45	0.00000108609949932	0.00000245205207101	−0.0006144058	0.020186718
46	−0.00000622164710970	0.00021113849940800	−0.0023352259	0.023344489
47	0.00002478668288580	−0.00068728212781100	0.0053024871	0.007762005

The age based values of $\phi(x,t)$ are

				t			
x	2017	2018	2019	2020	2021	2022	202
42	0.028763796	0.0306985840816	0.0315277905271	0.0314168675271	0.0305312672724	0.0290364419539	0.02
43	0.013387987	0.0133494893141	0.0133803882511	0.0134691437633	0.0136042158028	0.0137740643219	0.01
44	0.012284496	0.0133701859176	0.014205243481	0.0148181681962	0.0152374595690	0.0154916171055	0.01
45	0.020186718	0.0181316689068	0.0165493762541	0.0153861824119	0.0145884297505	0.0141024606398	0.01
46	0.023344489	0.0225305629052	0.0216810727055	0.0208135378079	0.0199454776195	0.0190944115474	0.01
47	0.007762005	0.0098218759479	0.0114801079633	0.0127779161143	0.0137565154685	0.0144571210939	0.01

the cohort-based values $\phi(x+t-2017, t)$ are:

				t		
x	2017	2018	2019	2020	2021	2022
42	0.028763796	0.0276794181105	0.0264539143812	0.0251269384206	0.0237381438374	0.022327
0.0209337132380						
43	0.013387987	0.0121020933870	0.0112840700959	0.0108704022485	0.0107975749669	0.011002
0.0114203825882						
44	0.012284496	0.0120808545406	0.0120384504724	0.0121320611607	0.0123364639707	0.012626
0.0129767554168						
45	0.020186718	0.0195758503516	0.0189764034043	0.0183948937551	0.0178378380011	0.017311
0.0168231545664						
46	0.023344489	0.0212141799523	0.0194688180208	0.0180710733227	0.0169836159755	0.016169
0.0155902438030						
47	0.007762005	0.0124019966551	0.0158161441519	0.0181531675876	0.0195617870597	0.020190
0.0201886945021						

Taking the average values for $\phi(x,t)$ gives

				t		
x	2017	2018	2019	2020	2021	2022
42	0.028763796					
43	0.013387987	0.0205144537123				
44	0.012284496	0.0127361396523	0.0203295789311			
45	0.020186718	0.0151062617237	0.013916723175	0.0202565604163		
46	0.023344489	0.0210532066284	0.016859761589	0.0158419700282	0.0218418107285	
47	0.007762005	0.0155180279501	0.0152282556838	0.0124549886375	0.0122770452177	0.0183921526671

We get the factors as the cumulative products of $1-\phi(x,t)$:

				t		
x	2017	2018	2019	2020	2021	2022
42	1	0.971236204				
43	1	0.986612013	0.966372206528			
44	1	0.987715504	0.975135821405	0.955311720755		
45	1	0.979813282	0.965011966121	0.951582161729	0.932306380179	
46	1	0.976655511	0.956093780723	0.939974267522	0.92508322335	0.904877730678
47	1	0.992237995	0.976840418061	0.961964842412	0.949983581231	0.938320589848

This gives

t	$q(42+t, 2017+t)$	${}_tp_{42}q(42+t, 2017+t)$
0	$1 \times 0.000678 = 0.000678$	0.000678
1	$0.986612013 \times 0.0007354987 = 0.000725651852966$	0.00072515986101
2	$0.975135821405 \times 0.0007991292 = 0.000779259508851$	0.000778166083187
3	$0.951582161729 \times 0.0008679190 = 0.000825896238226$	0.000824094689335
4	$0.92508322335 \times 0.0009418973 = 0.000871333390349$	0.000868714666999

This gives $A^1_{42:\overline{5}|} = 0.000678(1.05)^{-1} + 0.00072515986101(1.05)^{-2} + 0.000778166083187(1.05)^{-3} + 0.000824094689335(1.05)^{-4}$ $0.000868714666999(1.05)^{-5} = 0.00333431018998$

Standard Questions

4. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.6 \qquad \sigma_k = 1.2 \qquad K_{2017} = -6.24$$

And the following values of α_x and β_x:

x	α_x	β_x
41	-4.316805	0.2697564
42	-4.330498	0.1998375
43	-4.349431	0.2408687
44	-4.372390	0.2650377
45	-4.397883	0.1142745
46	-4.423151	0.1246374
47	-4.431606	0.2082677

The insurance company simulates the following values of Z_t:

-1.2365624 0.1837002 1.2881093 1.0537143 -0.9344071 0.0940466

Using these simulated values, calculate the probability that a life aged exactly 41 at the start of 2017 survives for 6 years.

We have that $K_{t+1} = K_t + c + \sigma_k Z_t$. With the simulated values, this gives

t	Z_{t-1}	$c + \sigma_k Z_{t-1}$	K_t
		-6.24	
2017	-1.2365624	-2.08387488	-8.32387488
2018	0.1837002	-0.37955976	-8.70343464
2019	1.2881093	0.94573116	-7.75770348
2020	1.0537143	0.66445716	-7.09324632
2021	-0.9344071	-1.72128852	-8.81453484
2022	0.0940466	-0.48714408	-9.30167892

Now we have $\log(m(x,t)) = \alpha_x + \beta_x K_t$, which gives the following values for $m(x,t)$:

x	2017	2018	2019	2020	2021	2022
41	0.00247854165031	0.00141274095528	0.00127525107764	0.00164585518092	0.0019689525235	0.00123759885
42	0.00378206924431	0.00249386803446	0.00231170340862	0.00279261158450	0.0031891699128	0.00226094456
43	0.00287284719202	0.00173909238213	0.00158714894721	0.00199318991313	0.0023391398097	0.00154523921
44	0.00241459128105	0.00138989002555	0.00125687311261	0.00161491349434	0.0019258888257	0.00122040312
45	0.00603031590938	0.00475247549438	0.00455074810083	0.00507012054093	0.0054700901789	0.00449333738
46	0.00551166581355	0.00425093158766	0.00405451394337	0.00456174071880	0.0049556111493	0.00399875704

We then use the approximation $q_x \approx 1 - e^{-m_x}$ to get the following values for $q(x,t)$:

x	2017	2018	2019	2020	2021	2022	
41	0.002475472602	0.001411743507	0.001274438291	0.001644501504	0.001967015408	0.001236833343	0.0010
42	0.003774926228	0.002490760929	0.002309033480	0.002788715872	0.003184089912	0.002258390550	0.0020
43	0.002868724515	0.001737581037	0.001585890092	0.001991204829	0.002336406154	0.001544045949	0.0013
44	0.0024116785	0.001388924576	0.001256083578	0.001613610223	0.001924035492	0.001219658741	0.0010
45	0.006012170048	0.004741200351	0.004540409136	0.005057289174	0.005455156478	0.004483257443	0.0042
46	0.005496504451	0.004241909167	0.004046305499	0.004551351783	0.004943352367	0.003990772663	0.0037

The probability that the life survives for 6 years is then given by

$$(1-0.002475472602)(1-0.002490760929)(1-0.001585890092)(1-0.001613610223)(1-0.005455156478)(1-0.003990772663)$$