ACSC/STAT 4720, Life Contingencies II
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Homework Sheet 6
Model Solutions

Basic Questions

1. *An individual aged 45 has a current salary of $67,000. The salary scale is $s_y = 1.05^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 65.*

   The final 3 years are the years at the start of which the individual is aged 62, 63 and 64. The final average salary is therefore

$$67000\frac{(1.05)^{17}+(1.05)^{18}+(1.05)^{19}}{3} = \$161,371.46$$

2. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 70% of final average salary for an employee who enters the plan at exact age 26, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 70% of the life annuity.*
   - *At age 65, the employee is married to someone aged 62.*
   - *The salary scale is $s_y = 1.06^y$.*
   - *Mortalities are independent and given by $\mu_x = 0.0000017(1.101)^x$.*
   - *A fixed percentage of salary is payable monthly in arrear.*
   - *Contributions earn an annual rate of 5%.*
   - *The value of the life annuity is based on $\delta = 0.04$.*

   Calculate the percentage of salary payable monthly to achieve the target replacement rate under these assumptions. [You may use numerical integration to compute the value of the annuities.]

   The survival probability for an individual aged 65 is

$$e^{-\int_{65}^{65+t}0.0000017(1.101)^x\,dx} = e^{-\int_{65}^{65+t}0.0000017e^{x\log(1.101)}\,dx} = e^{-\left[\frac{0.0000017}{\log(1.101)}e^{x\log(1.101)}\right]_{65}^{65+t}} = e^{-0.00919103579853\left((1.101)^t-1\right)}$$

   We have

$$\overline{a}_{65} = \int_0^\infty e^{-0.04t}e^{-0.00919103579853\left((1.101)^t-1\right)}\,dt$$

   Evaluating this numerically gives

$$\overline{a}_{65} = 19.91366$$

   Similarly the survival probability for an individual aged 62 is

$$e^{-\int_{62}^{62+t}0.0000017(1.101)^x\,dx} = e^{-\int_{62}^{62+t}0.0000017e^{x\log(1.101)}\,dx} = e^{-\left[\frac{0.0000017}{\log(1.101)}e^{x\log(1.101)}\right]_{62}^{62+t}} = e^{-0.00688656261735\left((1.101)^t-1\right)}$$

The probability that t years after retirement the individual is dead and the spouse is alive is therefore

$$e^{-0.00688656261735\left((1.101)^t-1\right)}\left(1-e^{-0.00919103579853\left((1.101)^t-1\right)}\right)$$

$$\overline{a}_{65|62}=\int_0^\infty e^{-0.04t}e^{-0.00688656261735\left((1.101)^t-1\right)}\left(1-e^{-0.00919103579853\left((1.101)^t-1\right)}\right)\,dt$$

Evaluating this numerically gives

$$\overline{a}_{65|62}=1.714545$$

If the final average salary is F, then the cost of these annuities is $(19.91366\times 0.7+1.714545\times 0.7\times 0.7)F=14.77968905F$. If the employee starts with salary S at age 26, then the employee's final average salary is $F=\frac{1.06^{36}+1.06^{37}+1.06^{38}}{3}S=8.64586382223S$.

The accumulated value should therefore be $14.77968905\times 8.64586382223S=127.783178861S$. Let M be the employee's first month's salary. If the employee contributed all their salary into the pension plan, the accumulated value would be $M\frac{1.06^{39}-1.05^{39}}{1.06^{\frac{1}{12}}-1.05^{\frac{1}{12}}}=3779.49972353M$. We also have that the first year's salary is $S=\frac{1.06-1}{1.06^{\frac{1}{12}}-1}M=12.3265283302M$. Therefore, the accumulated value of the pension fund would be $\frac{3779.49972353}{12.3265283302}S=306.615100561S$

The proportion of salary payable to the pension fund is therefore

$$\frac{127.783178861}{306.615100561}=41.68\%$$

3. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
30	*1.000000*	*39*	*1.350398*	*48*	*1.845766*	*57*	*2.553877*
31	*1.033333*	*40*	*1.397268*	*49*	*1.912422*	*58*	*2.649694*
32	*1.067933*	*41*	*1.445983*	*50*	*1.981785*	*59*	*2.749515*
33	*1.103853*	*42*	*1.496620*	*51*	*2.053975*	*60*	*2.853522*
34	*1.141149*	*43*	*1.549263*	*52*	*2.129115*	*61*	*2.961903*
35	*1.179879*	*44*	*1.604000*	*53*	*2.207337*	*62*	*3.074855*
36	*1.220103*	*45*	*1.660921*	*54*	*2.288777*	*63*	*3.192585*
37	*1.261887*	*46*	*1.720122*	*55*	*2.373580*	*64*	*3.315310*
38	*1.305295*	*47*	*1.781702*	*56*	*2.461894*	*65*	*3.443256*

An employee aged 46 and 2 months has 8 years of service, and a current salary of \$83,000 (for the coming year). She has a defined benefit pension plan with $\alpha=0.015$ and S_{Fin} is the average of her last 3 years' salary. The employee's mortality is given by $\mu_x=0.00000104(1.111)^x$. The pension benefit is payable monthly in advance. The interest rate is $i=0.06$. [This gives $\ddot{a}^{(12)}_{65}=14.98951$.] Calculate the EPV of the accrued benefit under the assumption that the employee retires at age 65.

Using linear interpolation, we get

$$s_{46\frac{2}{12}}=\frac{2}{12}s_{47}+\frac{10}{12}s_{46}=\frac{2\times 1.781702+10\times 1.720122}{12}=1.73038533333$$

The estimated final average salary for this employee is therefore

$$\frac{3.074855+3.192585+3.315310}{3\times 1.73038533333}\times 83000=\$153,216.02$$

The accrued annual annuity payment is therefore $153216.02 \times 0.015 \times 8 = \$18,385.92$.

The value of the pension at time of retirement is therefore

$$14.98951 \times 18385.92 = \$275,595.93$$

We need to discount this by 18 years 10 months, so the EPV is

$$275595.93(1.06)^{-18\frac{10}{12}} = \$91,976.95$$

Standard Questions

4. *An employee aged 58 has been working with a company for 23 years. The employee's salary last year was \$94,000. The salary scale is the same as for Question 3. The service table is given below:*

t	${}_tp^{(00)}$	1	2	3
0	10000.00	48.48	0	6.48
1	9945.03	45.29	0	7.29
2^-	9892.45		1187.53	
2	8704.92	25.72	142.14	6.86
3	8530.20	21.69	128.54	7.56
4^-	8372.40		1426.64	
4	6945.76	12.81	455.53	6.95
5	6470.47	10.83	757.69	7.29
6	5694.66	9.16	416.44	7.23
7^-	5261.82		5311.82	

Mortality follows a Gompertz model with $B = 0.0000012$ and $C = 1.1$. If the member withdraws before age 60, he receives a defered pension starting from age 65, with 2% COLA. The death benefit of the plan is three times the employee's final average salary if the employee is still working at the time of death. If the employee has withdrawn, the death benefit is three times final average salary with COLA of 2%. The accrual rate for the pension is 0.02. Pension payments are made annually in advance. The interest rate is $i = 0.04$.

Calculate the EPV of the accrued benefit. [You may assume that events happen in the middle of each year.]

You are given the following values:

x	$\ddot{a}_x$
60	20.85185
60.5	20.75652
61.5	20.56113
62	20.46258
62.5	20.35935
63.5	20.15106
64.5	19.93613
65	19.83053

For an individual who withdraws at age x with final average salary F, the expected present value of death benefit paid during the withdrawl period is

$$3F\int_0^{65-x}\mu_{x+t}e^{-\int_0^t \mu_{x+s}\,ds}(1.02)^t(1.04)^{-t}\,dt = 3F\int_0^{65-x}0.0000012(1.1)^{x+t}e^{-\frac{0.0000012}{\log(1.1)}(1.1^{x+t}-1.1^x)}(1.02)^t(1.04)^{-t}\,dt$$

$$= 3F0.0000012(1.1)^x\int_0^{65-x}e^{-\frac{0.0000012(1.1)^x}{\log(1.1)}(1.1^t-1)}\left(\frac{1.122}{1.04}\right)^t\,dt$$

The EPV of death benefits at time of withdrawl at each age of withdrawl is therefore.

Age	EPV
58.5	$0.007972536 \times 0.46F$
59.5	$0.007124941 \times 0.46F$
60	$0.006657889 \times 0.46F$
60.5	$0.006160001 \times 0.46F$
61.5	$0.005065258 \times 0.46F$
62	$0.004464971 \times 0.46F$
62.5	$0.003826975 \times 0.46F$
63.5	$0.002430006 \times 0.46F$
64.5	$0.000857642 \times 0.46F$

We calculate the following:

Age at exit	Final ave. sal.	Withdrawn Surv prob	COLA to age 65	Exp. D. Ben.	Exp Def pension	Exp Pen Ben	Exp W. D. Ben.	Exp Tot Ben	Exp Disc Tot. Ben.
58.5	89013.59	0.9971529	1.137368	173.04	7521.41	0.00	1.58	7696.04	7546.58
59.5	92346.75	0.9974843	1.115067	201.96	7435.02	0.00	1.37	7638.35	7201.94
60	94047.35	0.9976623	1.104081	0.00	0.00	107125.67	0.00	107125.67	99043.70
60.5	95819.04	0.9978490	1.093203	197.20	4468.61	13004.11	0.70	17670.61	16020.21
61.5	99436.71	0.9982503	1.071768	225.52	3989.00	12088.98	0.50	16304.01	14212.74
62	101282.68	0.9984658	1.061208	0.00	0.00	136009.05	0.00	136009.05	116261.11
62.5	103206.30	0.9986919	1.050752	215.18	2494.24	44029.62	0.23	46739.28	39177.07
63.5	107134.67	0.9991779	1.030150	234.30	2232.99	75244.94	0.13	77712.36	62633.51
64.5	111228.99	0.9997127	1.009950	241.25	2000.35	42478.49	0.04	44720.14	34656.64
65	113318.56	1.0000000	1.000000	0.00	0.00	549081.15	0.00	549081.15	417256.55

As an example, the calculation of the row $x = 60.5$ is calculated as follows:

$$94000\frac{\frac{2.553877+2.649694}{2}+\frac{2.649694+2.749515}{2}+\frac{2.749515+2.853522}{2}}{3\times 2.649694} = \$95,819.04$$

$$e^{-\int_{60.5}^{65}0.0000012(1.1)^x\,dx} = e^{-\frac{0.0000012}{\log(1.1)}(1.1^{65}-1.1^{60.5}} = 0.9978490$$

$$(1.02)^{4.5} = 1.093203$$

$$3\times 95819.04\times\frac{6.86}{10000} = 197.20$$

$$23 \times 0.02 \times 95819.04 \times 1.093203 \times 0.9978490 \times 19.83053 \times \frac{25.72}{10000} = 4468.61$$
$$23 \times 0.02 \times 95819.04 \times 20.75652 \times \times (1.04)^{-4.5} \frac{142.14}{10000} = 13004.11$$
$$0.006160001 \times 0.46 \times 95819.04 \times \frac{25.72}{10000} = 0.70$$
$$197.20 + 4468.61 + 13004.11 + 0.70 = 17670.61$$
$$17670.61(1.04)^{-2.5} = 16020.21$$

So the total EPV of the accrued benefit is \$814,010.06.

5. *An individual aged 43 has 4 years of service, and last year's salary was \$74,000. The salary scale is $s_y = 1.05^y$. The accrual rate is 0.01. The interest rate is $i = 0.05$. There is no death benefit, and no exits other than death or retirement at age 65. The pension benefit is payable annually in advance. Mortality follows a Gompertz law with $B = 0.000004$ and $C = 1.09$. You are given that $\ddot{a}_{65} = 18.04168$. Calculate this year's employer contribution to the plan using*

   (a) The projected unit method.

   Under the projected unit method, the estimated final average salary is $74000\frac{1.05^{20}+1.05^{21}+1.05^{22}}{3} = \$206,324.85$. If the employee retires at age 65, the EPV of the accrued pension at that time is therefore

$$18.04168 \times 206324.851715 \times 0.01 \times 4 = \$148,897.88$$

   The current EPV is therefore

$$148897.878028(1.05)^{-22} e^{-\frac{0.000004}{\log(1.09)}(1.09^{65}-1.09^{43})} = 50359.82$$

   In one year's time, the current EPV of the pension will be

$$18.04168 \times 206324.851715 \times 0.01 \times 5(1.05)^{-21} e^{-\frac{0.000004}{\log(1.09)}(1.09^{65}-1.09^{44})} = \$66108.49$$

   The expected present value of this is

$$66108.4921856(1.05)^{-1} e^{-\frac{0.000004}{\log(1.09)}(1.09^{44}-1.09^{43})} = 62949.7713518$$

   The employer contribution is therefore

$$62949.77 - 50359.82 = 12,589.95$$

   (b) The traditional unit method.

   Using the traditional unit method the current final average salary is $74000\frac{1.05^{-2}+1.05^{-1}+1}{3} = 70532.1239607$. The current EPV is therefore

$$18.04168 \times 70532.1239607 \times 0.01 \times 4(1.05)^{-22} e^{-\frac{0.000004}{\log(1.09)}(1.09^{65}-1.09^{43})} = 17215.4969772$$

   After another year, the employee's final average salary will be $74000\frac{1.05^{-1}+1+1.05}{3} = 74058.73$. The current EPV of the accrued benefits at this point is therefore

$$18.04168 \times 74058.7301585 \times 0.01 \times 5(1.05)^{-21} e^{-\frac{0.000004}{\log(1.09)}(1.09^{65}-1.09^{44})} = \$23,729.14$$

   This year's employer contribution is the change in EPV, which is $23729.1385079e^{-\frac{0.000004}{\log(1.09)}(1.09^{44}-1.09^{43})}(1.05)^{-1} - 17215.50 = \$5,379.84$.