# ACSC/STAT 4720, Life Contingencies II
Fall 2018
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Homework Sheet 2
Model Solutions

## Basic Questions

1. *The following is a standard multiple decrement table giving probabilities of death and surrender for a life insurance policy:*

| $x$ | $l_x$ | $d_x^{(1)}$ | $d_x^{(2)}$ |
|---|---|---|---|
| 46 | 10000.00 | 43.90 | 1.20 |
| 47 | 9954.90 | 43.69 | 1.24 |
| 48 | 9909.96 | 43.50 | 1.27 |
| 49 | 9865.19 | 43.29 | 1.31 |
| 50 | 9820.59 | 43.09 | 1.35 |

*A life who is in poor health has the following lifetable.*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| 46 | 10000.00 | 133.11 |
| 47 | 9866.89 | 146.03 |
| 48 | 9720.86 | 159.98 |
| 49 | 9560.88 | 174.95 |
| 50 | 9385.93 | 190.98 |

*Use this lifetable and the standard multiple decrement table to produce a multiple decrement table for this life, assuming that this life has standard surrender probabilities, using:*

*(a) UDD in the multiple decrement table.*

Recall that for UDD in the multiple decrement table, we have ${}_tp^{01} = tp^{01}$ and ${}_tp^{02} = tp^{02}$. This gives us $\mu_{x+t}^{01} = \frac{p^{01}}{1-t(p^{01}+p^{02})}$, so the individual decrement probability is

$$p^1 = e^{-\int_0^1 \mu_{x+t}^1\,dt} = e^{-\int_0^1 \frac{p^{01}}{1-t(p^{01}+p^{02})}\,dt}$$

And we have

$$\begin{aligned}
\int_0^1 \frac{p^{01}}{1-t(p^{01}+p^{02})}\,dt &= \frac{p^{01}}{p^{01}+p^{02}}\int_0^1 \frac{1}{\frac{1}{p^{01}+p^{02}}-t}\,dt \\
&= \frac{p^{01}}{p^{01}+p^{02}}\int_{\frac{1}{p^{01}+p^{02}}-1}^{\frac{1}{p^{01}+p^{02}}} \frac{1}{u}\,du \\
&= \frac{p^{01}}{p^{01}+p^{02}}\left[\log(u)\right]_{\frac{1}{p^{01}+p^{02}}-1}^{\frac{1}{p^{01}+p^{02}}} \\
&= \frac{p^{01}}{p^{01}+p^{02}}\log\left(\frac{\frac{1}{p^{01}+p^{02}}}{\frac{1}{p^{01}+p^{02}}-1}\right) \\
&= \frac{p^{01}}{p^{01}+p^{02}}\log\left(\frac{1}{1-p^{01}-p^{02}}\right)
\end{aligned}$$

This gives us

$$p^1 = e^{-\frac{p^{01}}{p^{01}+p^{02}}\log\left(\frac{1}{1-p^{01}-p^{02}}\right)} = \left(\frac{1}{1-p^{01}-p^{02}}\right)^{-\frac{p^{01}}{p^{01}+p^{02}}} = \left(1-p^{01}-p^{02}\right)^{\frac{p^{01}}{p^{01}+p^{02}}}$$

Similarly

$$p^2 = \left(1-p^{01}-p^{02}\right)^{\frac{p^{02}}{p^{01}+p^{02}}}$$

Now from the multiple decrement table, the individual decrement probabilites for decrement 1 are given by:

| age | $p^1$ |
|---|---|
| 46 | $\left(\frac{9954.90}{10000.00}\right)^{\frac{43.90}{43.90+1.20}} = 0.995609736193$ |
| 47 | $\left(\frac{9909.96}{9954.90}\right)^{\frac{43.69}{43.69+1.24}} = 0.995609955851$ |
| 48 | $\left(\frac{9865.19}{9909.96}\right)^{\frac{43.50}{43.50+1.27}} = 0.995610195029$ |
| 49 | $\left(\frac{9820.59}{9865.19}\right)^{\frac{43.29}{43.29+1.31}} = 0.995611551454$ |
| 50 | $\left(\frac{9776.15}{9820.59}\right)^{\frac{43.09}{43.09+1.35}} = 0.995611977862$ |

Conversely, given the single decrement probabilities $p^1$ and $p^2$, we need to find $p^{01}$ and $p^{02}$ which satisfy

$$\begin{aligned}
\left(1-p^{01}-p^{02}\right)^{\frac{p^{01}}{p^{01}+p^{02}}} &= p^1 \\
\left(1-p^{01}-p^{02}\right)^{\frac{p^{02}}{p^{01}+p^{02}}} &= p^2 \\
\left(1-p^{01}-p^{02}\right) &= p^1p^2 \\
\frac{p^{01}}{p^{01}+p^{02}} &= \frac{\log(p^1)}{\log(p^1p^2)} \\
p^{01} &= (1-p^1p^2)\frac{\log(p^1)}{\log(p^1p^2)}
\end{aligned}$$

This gives us the new values for $p^{01}$ and $p^{02}$

| age | $\frac{\log(p^1)}{\log(p^1p^2)}$ | $p^{01}$ | $p^{02}$ |
|---|---|---|---|
| 46 | $\frac{\log(0.995609736193)}{\log(0.995609736193\times\frac{9866.89}{10000.00})} = 0.247182669092$ | 0.00436100057506 | 0.0132818244299 |
| 47 | $\frac{\log(0.995609955851)}{\log(0.995609955851\times\frac{9720.86}{9866.89})}$ 0.227842299219 | 0.00435750090603 | 0.014767573415 |
| 48 | $\frac{\log(0.995610195029)}{\log(0.995610195029\times\frac{9560.88}{9720.86})} = 0.209560494429$ | 0.00435360918373 | 0.0164213426773 |
| 49 | $\frac{\log(0.995611551454)}{\log(0.995611551454\times\frac{9385.93}{9560.88})} = 0.192341600439$ | 0.00434820332555 | 0.0182584679074 |
| 50 | $\frac{\log(0.995611977862)}{\log(0.995611977862\times\frac{9194.95}{9385.93})} = 0.176224195048$ | 0.00434325931301 | 0.020302955197 |

The new multiple decrement table is therefore

| $x$ | $l_x$ | $d_x^{(1)}$ | $d_x^{(2)}$ |
|---|---|---|---|
| 46 | 10000.00 | 43.61 | 132.82 |
| 47 | 9823.57 | 42.80 | 145.07 |
| 48 | 9635.70 | 41.95 | 158.23 |
| 49 | 9435.52 | 41.03 | 172.28 |
| 50 | 9222.22 | 40.05 | 187.24 |

*(b) UDD in the independent decrements.*

Using UDD in the independent decrements, we get ${}_tp^1 = 1 - tq^1$, so $\mu^1_{x+t} = \frac{q^1}{1-tq^1}$. This gives

$$ {}_tp^{00} =_t p^1_t p^2 = (1 - tq^1)(1 - tq^2) $$

and thus

$$ \begin{aligned} p^{01} &= \int_0^1 (1 - tq^1)(1 - tq^2)\frac{q^1}{(1 - tq^1)}\,dt \\ &= q^1\int_0^1 (1 - tq^2)\,dt \\ &= q^1\left(1 - \frac{q^2}{2}\right) \end{aligned} $$

Given a multiple decrement table, we obtain the single decrement probabilities by solving

$$ \begin{aligned} q^1\left(1 - \frac{q^2}{2}\right) &= p^{01} \\ q^2\left(1 - \frac{q^1}{2}\right) &= p^{02} \\ q^1 - q^2 &= p^{01} - p^{02} \\ p^1p^2 &= p^{00} \\ p^1(p^1 + p^{01} - p^{02}) &= p^{00} \\ p^1 &= \frac{p^{02} - p^{01} + \sqrt{(p^{01} - p^{02})^2 + 4p^{00}}}{2} \end{aligned} $$

Now from the multiple decrement table, the individual decrement probabilites for decrement 1 are given by:

| age | $p^1$ |
|---|---|
| 46 | $1-\frac{1}{2}\left(\frac{1.20-43.90}{10000.00}+\sqrt{\left(\frac{43.90-1.20}{10000.00}\right)^2+4\frac{9954.90}{10000.00}}\right)=0.004390263995$ |
| - 47 | $1-\frac{1}{2}\left(\frac{1.24-43.69}{9954.90}+\sqrt{\left(\frac{43.69-1.24}{9954.90}\right)^2+4\frac{9909.96}{9954.90}}\right)=0.004389570815$ |
| 48 | $1-\frac{1}{2}\left(\frac{1.27-43.50}{9909.96}+\sqrt{\left(\frac{43.50-1.27}{9909.96}\right)^2+4\frac{9865.19}{9909.96}}\right)=0.00438980517$ |
| - 49 | $1-\frac{1}{2}\left(\frac{1.31-43.29}{9865.19}+\sqrt{\left(\frac{43.29-1.31}{9865.19}\right)^2+4\frac{9820.59}{9865.19}}\right)=0.004388448755$ |
| 50 | $1-\frac{1}{2}\left(\frac{1.35-43.09}{9820.59}+\sqrt{\left(\frac{43.09-1.35}{9820.59}\right)^2+4\frac{9776.15}{9820.59}}\right)=0.00438802235$ |

Conversely, we use the formulae

$$p^{01}=q^1\left(1-\frac{q^2}{2}\right)$$
$$p^{02}=q^2\left(1-\frac{q^1}{2}\right)$$

To obtain the multiple decrement table:

| age | $p^{01}$ | $p^{02}$ |
|---|---|---|
| 46 | $0.004390263995\left(1-\frac{133.11}{2\times 10000.00}\right)=0.00436104459298$ | $\frac{133.11}{10000.00}\left(1-\frac{0.004390263995}{2}\right)=0.013281780598$ |
| 47 | $0.004389570815\left(1-\frac{146.03}{2\times 9866.89}\right)=0.00435708798474$ | $\frac{146.03}{9866.89}\left(1-\frac{0.004389570815}{2}\right)=0.0147675200075$ |
| 48 | $0.00438980517\left(1-\frac{159.98}{2\times 9720.86}\right)=0.00435368279857$ | $\frac{159.98}{9720.86}\left(1-\frac{0.00438980517}{2}\right)=0.0164212692586$ |
| 49 | $0.004388448755\left(1-\frac{174.95}{2\times 9560.88}\right)=0.00434829768576$ | $\frac{174.95}{9560.88}\left(1-\frac{0.004388448755}{2}\right)=0.0182583737528$ |
| 50 | $0.00438802235\left(1-\frac{190.98}{2\times 9385.93}\right)=0.00434337975686$ | $\frac{190.98}{9385.93}\left(1-\frac{0.00438802235}{2}\right)=0.0203028349611$ |

The new multiple decrement table is therefore

| $x$ | $l_x$ | $d_x^{(1)}$ | $d_x^{(2)}$ |
|---|---|---|---|
| 46 | 10000.00 | 43.61 | 132.82 |
| 47 | 9823.57 | 42.80 | 145.07 |
| 48 | 9635.70 | 41.95 | 158.23 |
| 49 | 9435.52 | 41.03 | 172.28 |
| 50 | 9222.22 | 40.06 | 187.24 |

2. *The mortalities for a husband and wife (whose lives are assumed to be independent) aged 52 and 68 respectively, are given in the following tables:*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *52* | *10000.00* | *40.92* |
| *53* | *9959.08* | *45.30* |
| *54* | *9913.78* | *50.14* |
| *55* | *9863.65* | *55.46* |
| *56* | *9808.19* | *61.31* |
| *57* | *9746.88* | *67.74* |
| *58* | *9679.14* | *74.79* |
| *59* | *9604.35* | *82.52* |
| *60* | *9521.83* | *90.96* |
| *61* | *9430.87* | *100.17* |

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *68* | *10000.00* | *77.76* |
| *69* | *9922.24* | *84.48* |
| *70* | *9837.76* | *91.70* |
| *71* | *9746.06* | *99.47* |
| *72* | *9646.59* | *107.80* |
| *73* | *9538.79* | *116.71* |
| *74* | *9422.08* | *126.23* |
| *75* | *9295.85* | *136.36* |
| *76* | *9159.49* | *147.11* |
| *77* | *9012.38* | *158.49* |

*The interest rate is* $i = 0.06$.

*(a) They want to purchase a 10-year joint life insurance policy with a death benefit of \$200,000. Annual premiums are payable while both are alive. Calculate the net premium for this policy using the equivalence principle.*

We calculate $\ddot{a}_{52,68:\overline{10}|} = \frac{10000.00\times10000.00+9959.08\times9922.24(1.06)^{-1}+9913.78\times9837.76(1.06)^{-2}+9863.65\times9746.06(1.06)^{-3}+9808.19\times9646.59(1.}{}$
7.33852859693

This gives $A_{52,68:\overline{10}|} = 1-\frac{0.06}{1.06}\times7.33852859693 = 0.584611588853$, so $A^{1}_{52,68:\overline{10}|} = 0.584611588853-(1.06)^{-10}\times 0.885389\times0.933070 = 0.123304959694$. The premium is therefore $\frac{200000\times0.123304959694}{7.33852859693} = \$3,360.48$.

*(b) They want to purchase a 10-year last survivor insurance with a benefit of \$7,000,000. Premiums are payable while either life is alive. Calculate the net premium for this policy using the equivalence principle.*

We calculate $\ddot{a}_{\overline{52,68}:\overline{10}|} = \frac{10000.00\times10000.00+(10000\times10000-(10000-9959.08)\times(10000-9922.24))(1.06)^{-1}+(10000\times10000-(10000-9913.78)\times(10}{}$
7.79082911663

This gives $A^{1}_{\overline{52,68}:\overline{10}|} = 1-\frac{0.06}{1.06}\times7.79082911663 = 0.559009672644$, so $A^{1}_{\overline{52,68}:\overline{10}|} = 0.559009672644-(1.06)^{-10}\times (1-0.114611\times0.06693) = 0.004898294169$. The premium is therefore $\frac{7000000\times0.004898294169}{7.79082911663} = \$4,401.08$.

3. *A husband is 82; the wife is 74. Their lifetables while both are alive, and the lifetable for the wife if the husband is dead, are given below:*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *82* | *10000.00* | *260.05* |
| *83* | *9739.95* | *280.63* |
| *84* | *9459.31* | *301.97* |
| *85* | *9157.34* | *323.90* |
| *86* | *8833.45* | *346.17* |
| *87* | *8487.27* | *368.52* |
| *88* | *8118.75* | *390.58* |
| *89* | *7728.17* | *411.94* |
| *90* | *7316.24* | *432.09* |
| *91* | *6884.15* | *450.47* |

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *74* | *10000.00* | *29.56* |
| *75* | *9970.44* | *32.06* |
| *76* | *9938.39* | *34.76* |
| *77* | *9903.63* | *37.67* |
| *78* | *9865.95* | *40.82* |
| *79* | *9825.13* | *44.22* |
| *80* | *9780.91* | *47.89* |
| *81* | *9733.01* | *51.84* |
| *82* | *9681.17* | *56.10* |
| *83* | *9625.08* | *60.67* |

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *74* | *10000.00* | *67.05* |
| *75* | *9932.95* | *72.45* |
| *76* | *9860.49* | *78.25* |
| *77* | *9782.25* | *84.45* |
| *78* | *9697.80* | *91.08* |
| *79* | *9606.72* | *98.15* |
| *80* | *9508.57* | *105.69* |
| *81* | *9402.87* | *113.71* |
| *82* | *9289.17* | *122.21* |
| *83* | *9166.96* | *131.21* |

*Calculate the probability that the wife is alive in 10 years time. Use the UDD assumption for handling changes to the wife's mortality in the year of the husband's death.*

Recall that under the UDD assumption, if the husband dies at time $t$ and the probability of the wife's dying while the husband is alive is $q_a$, and the wife's probability of dying after the husband is dead is $q_d$, then the overall probability of the wife surviving is $(1-tq_a)\frac{1-q_d}{1-tq_d}$. Conditional on the husband dying during the year, the probability of the wife surviving is

$$\begin{aligned}\int_0^1 (1-tq_a)\frac{1-q_d}{1-tq_d}\,dt &= (1-q_d)\int_0^1 \frac{q_a}{q_d}+\frac{1-\frac{q_a}{q_d}}{1-tq_d}\,dt\\ &= (1-q_d)\left(\frac{q_a}{q_d}+\frac{q_d-q_a}{(q_d)^2}\left[-\log(1-tq_d)\right]_0^1\right)\\ &= (1-q_d)\left(\frac{q_a}{q_d}+\frac{q_a-q_d}{(q_d)^2}\log\left(1-q_d\right)\right)\end{aligned}$$

This gives the following probabilities:

| Year | $q_a$ | $q_d$ | P(wife survives given husband dies) |
|---|---|---|---|
| 1 | 0.002956 | 0.006705 | $(1-0.006705)\left(\frac{0.002956}{0.006705}+\frac{0.002956-0.006705}{0.006705^2}\log\left(1-0.006705\right)\right)=0.995165296392$ |
| 2 | 0.0032155 | 0.0072939 | $(1-0.0072939)\left(\frac{0.0032155}{0.0072939}+\frac{0.0032155-0.0072939}{0.0072939^2}\log\left(1-0.0072939\right)\right)=0.994740318575$ |
| 3 | 0.0034975 | 0.0079357 | $(1-0.0079357)\left(\frac{0.0034975}{0.0079357}+\frac{0.0034975-0.0079357}{0.0079357^2}\log\left(1-0.0079357\right)\right)=0.994277476882$ |
| 4 | 0.0038037 | 0.0086330 | $(1-0.0086330)\left(\frac{0.0038037}{0.0086330}+\frac{0.0038036-0.0086330}{0.0086330^2}\log\left(1-0.0086330\right)\right)=0.993774701724$ |
| 5 | 0.0041375 | 0.0093918 | $(1-0.0093918)\left(\frac{0.0041375}{0.0093918}+\frac{0.0041374-0.0093918}{0.0093918^2}\log\left(1-0.0093918\right)\right)=0.993227094715$ |
| 6 | 0.0045007 | 0.0102168 | $(1-0.0102168)\left(\frac{0.0045007}{0.0102168}+\frac{0.0045007-0.0102168}{0.0102168^2}\log\left(1-0.0102168\right)\right)=0.992631461418$ |
| 7 | 0.0048963 | 0.0111152 | $(1-0.0111152)\left(\frac{0.0048963}{0.0111152}+\frac{0.0048962-0.0111152}{0.0111152^2}\log\left(1-0.0111152\right)\right)=0.991982660432$ |
| 8 | 0.0053262 | 0.0120931 | $(1-0.0120931)\left(\frac{0.0053262}{0.0120931}+\frac{0.0053262-0.0120931}{0.0120931^2}\log\left(1-0.0120931\right)\right)=0.991276617809$ |
| 9 | 0.0057948 | 0.0131562 | $(1-0.0131562)\left(\frac{0.0057948}{0.0131562}+\frac{0.0057948-0.0131562}{0.013156^2}\log\left(1-0.0131562\right)\right)=0.990508284137$ |
| 10 | 0.0063033 | 0.0143134 | $(1-0.0143134)\left(\frac{0.0063033}{0.0143134}+\frac{0.0063033-0.0143134}{0.0143134^2}\log\left(1-0.0143134\right)\right)=0.98967241144$ |

We now calculate the wife's overall survival probability:

| Year | P(Husband dies) | P(W survives to start) | P(W survives) | P(W survives from end) | P(W survives) |
|---|---|---|---|---|---|
| 1 | 0.026005 | 1.000000 | 0.99516529639 | $\frac{9035.75}{9932.95}=0.909674366628$ | 0.0235417117596 |
| 2 | 0.028063 | 0.997044 | 0.99474031857 | $\frac{9035.75}{9860.49}=0.916359126169$ | 0.0255049132701 |
| 3 | 0.030197 | 0.993839 | 0.99427747688 | $\frac{9035.75}{9782.25}=0.923688313016$ | 0.0275621368362 |
| 4 | 0.032390 | 0.990363 | 0.99377470172 | $\frac{9035.75}{9697.80}=0.931731939203$ | 0.0297019029455 |
| 5 | 0.034617 | 0.986595 | 0.99322709471 | $\frac{9035.75}{9606.72}=0.94056556244$ | 0.0319055305041 |
| 6 | 0.036852 | 0.982513 | 0.99263146141 | $\frac{9035.75}{9508.57}=0.950274331472$ | 0.0341535932806 |
| 7 | 0.039058 | 0.978091 | 0.99198266043 | $\frac{9035.75}{9402.87}=0.960956601548$ | 0.036416409105 |
| 8 | 0.041194 | 0.973301 | 0.99127661780 | $\frac{9035.75}{9289.17}=0.9727187682$ | 0.0386601283828 |
| 9 | 0.043209 | 0.968117 | 0.99050828414 | $\frac{9035.75}{9166.96}=0.985686639846$ | 0.0408412517115 |
| 10 | 0.045047 | 0.962508 | 0.98967241144 | 1 | 0.0429103132804 |
| $>10$ | 0.688415 | 0.956441 | 1 | 1 | 0.658428331015 |

The total probability is therefore

$0.0235417117596+0.0255049132701+0.0275621368362+0.0297019029455+0.0319055305041+0.0341535932806+0.036416409105+0.0386601283828+0.0408412517115+0.658428331015=0.946715908811$

# Standard Questions

4. *The following is a multiple decrement table giving probabilities of surrender (decrement 1) and death (decrement 2) for a life insurance policy:*

| $x$ | $l_x$ | $d_x^{(1)}$ | $d_x^{(2)}$ |
|---|---|---|---|
| *51* | *10000.00* | *30.00* | *9.72* |
| *52* | *9960.28* | *29.85* | *10.56* |
| *53* | *9919.87* | *29.69* | *11.47* |
| *54* | *9878.71* | *29.53* | *12.45* |
| *55* | *9836.73* | *29.35* | *13.52* |
| *56* | *9793.86* | *29.16* | *14.69* |
| *57* | *9750.01* | *28.96* | *15.95* |
| *58* | *9705.10* | *28.76* | *17.31* |
| *59* | *9659.04* | *28.53* | *18.79* |
| *60* | *9611.71* | *28.29* | *20.40* |

*A life insurance policy pays a benefit of $280,000 at the end of the year of death. Premiums are payable at the beginning of each year. Calculate the premium for a 10-year policy sold to a life aged 51 if the interest rate is* $i = 0.07$.

We calculate

$$A^{02}_{51:\overline{10|}} = 0.000972(1.07)^{-1}+0.001056(1.07)^{-2}+0.001147(1.07)^{-3}+0.001245(1.07)^{-4}+0.001352(1.07)^{-5}+0.001469(1.07)^{-6}+$$

and

$$\ddot{a}^{00}_{51:\overline{10|}} = 1+0.996028(1.07)^{-1}+0.991987(1.07)^{-2}+0.987871(1.07)^{-3}+0.983673(1.07)^{-4}+0.979386(1.07)^{-5}+0.975001(1.07)^{-}$$

so the premium is $\frac{280000\times 0.00971950241384}{7.39147970569} = \$368.19$.

5. *A couple want to receive the following:*

   - *While both are alive, they would like to receive a pension of $90,000 per year.*
   - *If the husband is alive and the wife is not, they would like to receive a pension of $60,000 per year.*
   - *If the wife is alive and the husband is not, they would like to receive a pension of $50,000 per year.*
   - *When the husband dies: if the wife is still alive, they would like a death benefit of $30,000; otherwise, they would like a death benefit of $90,000.*
   - *When the wife dies: if the husband is still alive, they would like a death benefit of $160,000; otherwise, they would like a death benefit of $120,000.*

   *Construct a combination of insurance and annuity policies that achieve this combination of benefits.*

   There are a number of possible solutions. Here is one example:

   - A life annuity for the husband of $40,000 per year.
   - A life annuity for the wife of $30,000 per year.
   - A last survivor annuity of $20,000 per year.
   - A last survivor life insurance of $60,000.
   - A contingent insurance policy of $100,000 payable if the wife dies while the husband is still alive.

- A life insurance policy of $60,000 on the wife.
- A life insurance policy of $30,000 on the husband.

This meets the needs of the couple as shown in the following table:

| | Both alive | H alive W dead | W alive H dead | W dies H alive | W dies H dead | H dies W alive | H dies W dead |
|---|---|---|---|---|---|---|---|
| life annuity (H) | $40,000 | $40,000 | $0 | $0 | $0 | $0 | $0 |
| life annuity (W) | $30,000 | $0 | $30,000 | $0 | $0 | $0 | $0 |
| last survivor annuity | $20,000 | $20,000 | $20,000 | $0 | $0 | $0 | $0 |
| life insurance (W) | $0 | $0 | $0 | $60,000 | $60,000 | $0 | $0 |
| life insurance (H) | $0 | $0 | $0 | $0 | $0 | $30,000 | $30,000 |
| contingent insurance | $0 | $0 | $0 | $100,000 | $0 | $0 | $0 |
| last survivor insurance | $0 | $0 | $0 | $0 | $60,000 | $0 | $60,000 |
| Total | $90,000 | $60,000 | $50,000 | $160,000 | $120,000 | $30,000 | $90,000 |

6. *A husband aged 68 and wife aged 75 have the following transition intensities:*

$$\begin{aligned}
\mu_{xy}^{01} &= 0.0002y + 0.0004 \\
\mu_{xy}^{02} &= 0.0003x - 0.0006 \\
\mu_{xy}^{03} &= 0.0007 \\
\mu_{x}^{13} &= 0.0004x + 0.0007 \\
\mu_{y}^{23} &= 0.0004y + 0.0005
\end{aligned}$$

*They want to purchase a last survivor insurance, which will pay a benefit of $700,000 when the second life dies. Premiums are payable continuously while either life is alive. Force of interest is* $\delta = 0.05$.

*(a) Calculate the annual rate of continuous premium.*

We calculate $\overline{a}_{\overline{x,y}} = \overline{a}_{x,y}^{00} + \overline{a}_{x,y}^{01} + \overline{a}_{x,y}^{02}$.

We have

$$\begin{aligned}
{}_tp_{68,75}^{00} &= e^{-\int_0^t 0.0005s+0.0359\,ds} = e^{-0.00025t^2-0.0359t} \\
{}_{t-s}p_{68+s}^{11} &= e^{-\int_s^t 0.0004(68+u)+0.0007\,du} = e^{-0.0002(t^2-s^2)-(0.0007+0.0272)(t-s)} \\
{}_{t-s}p_{75+s}^{22} &= e^{-\int_s^t 0.0004(75+u)+0.0005\,du} = e^{-0.0002(t^2-s^2)-(0.0005+0.03)(t-s)}
\end{aligned}$$

$$
\begin{aligned}
\overline{a}^{00}_{68,75} &= \int_0^\infty e^{-0.05t} e^{-0.00025t^2+0.0359t}\,dt \\
&= \int_0^\infty e^{-0.00025t^2-0.0859t}\,dt \\
&= \int_0^\infty e^{-0.00025(t^2+343.6t)}\,dt \\
&= \sqrt{\frac{\pi}{0.00025}} e^{7.37881}\left(1-\Phi\left(171.8\sqrt{0.0005}\right)\right) \\
&= 10.97511
\end{aligned}
$$

Next we calculate

$$
\begin{aligned}
{}_tp^{01}_{68,75} &= \int_0^t e^{-0.00025s^2-0.0359s}(0.0002(75+s)+0.0004)e^{-0.0002(t^2-s^2)-0.0279(t-s)}\,ds \\
&= e^{-0.0002t^2-0.0279t}\int_0^t (0.0154+0.0002s)e^{-0.00005(s^2+160s)}\,ds \\
&= e^{-0.0002t^2-0.0279t}\int_0^t 0.0002(s+77)e^{-0.00005(s^2+160s)}\,ds \\
&= e^{-0.0002t^2-0.0279t}e^{0.32}\int_0^t 0.0002(s+77)e^{-0.00005(s+80)^2}\,ds \\
&= e^{-0.0002t^2-0.0279t}e^{0.32}\int_0^t (0.0002(s+80)-0.0006)e^{-0.00005(s+80)^2}\,ds \\
&= e^{-0.0002t^2-0.0279t}e^{0.32}\left(\left[-2e^{-0.00005(s+80)^2}\right]_0^t - 0.0006\int_0^t e^{-0.00005(s+80)^2}\,ds\right) \\
&= e^{-0.0002t^2-0.0279t}e^{0.32}\left(2\left(e^{-0.32}-e^{-0.00005(t+80)^2}\right)-0.06\sqrt{2\pi}\left(\Phi\left(\frac{t+80}{100}\right)-\Phi\left(0.8\right)\right)\right)
\end{aligned}
$$

Numerically integrating this gives

$$
\begin{aligned}
\overline{a}^{01} &= \int_0^\infty e^{-0.05t}{}_tp^{01}\,dt \\
&= 2.212355
\end{aligned}
$$

where we compute this integral numerically:

```
t<-(1:1000000)/1000
integral1<-2*(exp(-0.32)-exp(-0.00005*(t+80)^2))
integral2<-0.06*sqrt(2*pi)*(pnorm((t+80)/100)-pnorm(0.8))
tp01<-exp(0.32-0.0279*t-0.0002*t^2)*(integral1-integral2)
sum(exp(-0.05*t)*tp01)/1000
```

Similarly

$$
\begin{aligned}
{}_tp^{02}_{68,75} &= \int_0^t e^{-0.00025s^2-0.0359s}(0.0003(68+s)-0.0006)e^{-0.0002(t^2-s^2)-0.0305(t-s)}\,ds \\
&= e^{-0.0002t^2-0.0305t}\int_0^t (0.0198+0.0003s)e^{-0.00005(s^2+108s)}\,ds \\
&= e^{-0.0002t^2-0.0305t}\int_0^t 0.0003(s+66)e^{-0.00005(s^2+108s)}\,ds \\
&= e^{-0.0002t^2-0.0305t}e^{0.1458}\int_0^t 0.0003(s+66)e^{-0.00005(s+54)^2}\,ds \\
&= e^{-0.0002t^2-0.0305t}e^{0.1458}\int_0^t (0.0003(s+54)+0.0036)e^{-0.00005(s+54)^2}\,ds \\
&= e^{-0.0002t^2-0.0305t}e^{0.1458}\left(3\left[-e^{-0.00005(s+54)^2}\right]_0^t+0.0036\int_0^t e^{-0.00005(s+54)^2}\,ds\right) \\
&= e^{-0.0002t^2-0.0305t}e^{0.1458}\left(3\left(e^{-0.1458}-e^{-0.00005(t+54)^2}\right)+0.36\sqrt{2\pi}\left(\Phi\left(\frac{t+54}{100}\right)-\Phi\left(0.54\right)\right)\right)
\end{aligned}
$$

Numerically integrating this gives

$$
\begin{aligned}
\overline{a}^{02} &= \int_0^\infty e^{-0.05t}{}_tp^{01}\,dt \\
&= 2.818177
\end{aligned}
$$

where we compute this integral numerically:

```
t<-(1:1000000)/1000
integral1<-3*(exp(-0.1458)-exp(-0.00015*(t+36)^2))
integral2<-0.36*sqrt(2*pi)*(pnorm((t+54)/100)-pnorm(0.54))
tp02<-exp(0.1458-0.0305*t-0.0002*t^2)*(integral1+integral2)
sum(exp(-0.05*t)*tp02)/1000
```

This gives $\overline{a}_{\overline{68,75}} = 10.97511 + 2.212355 + 2.818177 = 16.005642$ We have

$$\overline{A}_{\overline{68,75}} = 1 - \delta\overline{a}_{\overline{68,75}} = 1 - 0.05 \times 16.005642 = 0.1997179$$

The annual rate of premium is therefore $\frac{700000\times 0.1997179}{16.005642} = \$8,734.58$.

*(b) Calculate the policy value after 5 years if both are alive.*

In a similar way to the previous question, we compute:

$$
\begin{aligned}
{}_tp^{00}_{73,80} &= e^{-\int_0^t 0.0005s+0.0384\,ds} = e^{-0.00025t^2-0.0384t} \\
{}_{t-s}p^{11}_{68+s} &= e^{-\int_s^t 0.0004(73+u)+0.0007\,du} = e^{-0.0002(t^2-s^2)-(0.0007+0.0292)(t-s)} \\
{}_{t-s}p^{22}_{80+s} &= e^{-\int_s^t 0.0004(80+u)+0.0005\,du} = e^{-0.0002(t^2-s^2)-(0.0005+0.032)(t-s)}
\end{aligned}
$$

We therefore get

$$\begin{aligned}
\overline{a}^{00}_{73,80} &= 10.69589 \\
{}_tp^{01}_{73,80} &= e^{-0.0002t^2-0.0299t}e^{0.36125}\left(2\left(e^{-0.36125}-e^{-0.00005(t+85)^2}\right)-0.06\sqrt{2\pi}\left(\Phi\left(\frac{t+85}{100}\right)-\Phi\left(0.85\right)\right)\right) \\
\overline{a}^{01}_{\overline{73,80}} &= 2.227388 \\
{}_tp^{02}_{68,75} &= e^{-0.0002t^2-0.0325t}e^{0.31205}\left(3\left(e^{0.31205}-e^{-0.00005(t+79)^2}\right)-0.24\sqrt{2\pi}\left(\Phi\left(\frac{t+79}{100}\right)-\Phi\left(0.79\right)\right)\right) \\
\overline{a}^{02}_{\overline{73,80}} &= 2.799057
\end{aligned}$$

This gives $\overline{a}_{\overline{73,80}} = 10.69589+2.227388+2.799057 = 15.722335$, so $\overline{A}_{\overline{73,80}} = 1-0.05\times 15.722335 = 0.21388325$,. The policy value after 5 years is therefore $700000 \times 0.21388325 - 8734.58 \times 15.722335 = \$12,390.28$.