MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Problem Set 9 — Zorn's Lemma

Problem 1 If $\langle P, \leqslant \rangle$ is a partially ordered set (=poset), then a ***chain*** is a subset $C \subseteq P$ such that for all $x, y \in C$, $x \leqslant y$ or $y \leqslant x$. Show that any poset has a maximal chain, i.e., a chain that is not a proper subset of any other chain.

Problem 2 Prove Zorn's lemma for posets: If $\langle P, \leqslant \rangle$ is a poset such that any chain has an upper bound in P, then P has a maximal element. Hint: use Problem 1.

Problem 3 Give a direct proof of the Well-Ordering Theorem from Zorn's Lemma. Hint: Consider the set

$$\mathscr{A} = \{R \subseteq A \times A \mid R \text{ is a well-order on some subset of } A.\}$$

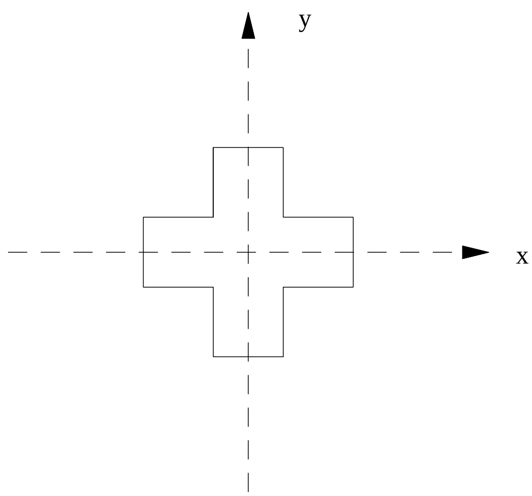
Problem 4 A subset of the Euclidean plane, $A \subseteq \mathbb{R} \times \mathbb{R}$, is called ***convex*** if for any two points $u, v \in A$, the line segment connecting u and v is contained in A. In other words, $A \subseteq \mathbb{R} \times \mathbb{R}$ is convex iff for all $u, v \in A$ and for all $\lambda \in \mathbb{R}$ with $0 \leqslant \lambda \leqslant 1$,

$$\lambda u + (1 - \lambda)v \in A.$$

(a) Prove that any subset X of the Euclidean plane has a maximal convex subset.

(b) Give at least three different examples of maximal convex subsets of the following subset of $\mathbb{R} \times \mathbb{R}$:

$$X = \{\langle x, y \rangle \mid (|x| \leqslant 1 \wedge |y| \leqslant 3) \vee (|x| \leqslant 3 \wedge |y| \leqslant 1)\}.$$



Problem 5 (Teichmüller-Tukey lemma) Assume that $\mathscr{A}$ is a nonempty set such that for every set B,

$$B \in \mathscr{A} \iff \text{every finite subset of } B \text{ is a member of } \mathscr{A}.$$

Show that $\mathscr{A}$ has a maximal element, i.e., an element that is not a subset of any other element of $\mathscr{A}$.